Exercise 7J

1 a l_1 has equation $\mathbf{r}=\begin{pmatrix}1\\3\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\5\end{pmatrix}$

l_2 has equation $\mathbf{r}=\begin{pmatrix}-1\\-3\\2\end{pmatrix}+\mu\begin{pmatrix}1\\1\\2\end{pmatrix}$

$\begin{pmatrix}1+\lambda\\3-\lambda\\5\lambda\end{pmatrix}=\begin{pmatrix}-1+\mu\\-3+\mu\\2+2\mu\end{pmatrix}$

$1+\lambda=-1+\mu\Rightarrow\lambda-\mu=-2$ **(1)**

$3-\lambda=-3+\mu\Rightarrow\lambda+\mu=6$ **(2)**

Adding **(1)** and **(2)** gives:

$\lambda-\mu+\lambda+\mu=-2+6$

$2\lambda=4$

$\lambda=2$

Substituting $\lambda=2$ into **(1)** gives:

$(2)-\mu=-2$

$\mu=4$

Checking that $\lambda=2$ and $\mu=4$ satisfy $5\lambda=2+2\mu$ gives:

$\text{LHS}=5(2)$

$\text{RHS}=2+2(4)$

$10=10$

So the lines do intersect.

Substituting $\lambda=2$ into $\begin{pmatrix}1\\3\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\5\end{pmatrix}$ gives:

$\begin{pmatrix}1\\3\\0\end{pmatrix}+2\begin{pmatrix}1\\-1\\5\end{pmatrix}=\begin{pmatrix}3\\1\\10\end{pmatrix}$

The point where the lines meet is (3, 1, 10)

b l_1 has equation $\mathbf{r}=\begin{pmatrix}3\\2\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\2\end{pmatrix}$

l_2 has equation $\mathbf{r}=\begin{pmatrix}4\\3\\0\end{pmatrix}+\mu\begin{pmatrix}-1\\1\\-1\end{pmatrix}$

$\begin{pmatrix}3+\lambda\\2+\lambda\\1+2\lambda\end{pmatrix}=\begin{pmatrix}4-\mu\\3+\mu\\-\mu\end{pmatrix}$

$3+\lambda=4-\mu\Rightarrow\lambda+\mu=1$ **(1)**

$2+\lambda=3+\mu\Rightarrow\lambda-\mu=1$ **(2)**

Adding **(1)** and **(2)** gives:

$\lambda+\mu+\lambda-\mu=1+1$

$2\lambda=2$

$\lambda=1$

Substituting $\lambda=1$ into **(1)** gives:

$1+\mu=1$

$\mu=0$

Checking that $\lambda=\frac{2}{3}$ and $\mu=\frac{1}{3}$ satisfy $1+\lambda=-\mu$ gives:

$\text{LHS}=1+2(1)$

$\text{RHS}=-(0)$

$3\neq0$

So the lines do not intersect.

1 c l_1 has equation $\mathbf{r} = \begin{pmatrix}1\\3\\5\end{pmatrix} + \lambda\begin{pmatrix}2\\3\\1\end{pmatrix}$

l_2 has equation $\mathbf{r} = \begin{pmatrix}1\\5/2\\5/2\end{pmatrix} + \mu\begin{pmatrix}1\\1\\-2\end{pmatrix}$

$\begin{pmatrix}1+2\lambda\\3+3\lambda\\5+\lambda\end{pmatrix} = \begin{pmatrix}1+\mu\\5/2+\mu\\5/2-2\mu\end{pmatrix}$

$1+2\lambda = 1+\mu \Rightarrow 2\lambda - \mu = 0$ **(1)**

$3+3\lambda = \frac{5}{2}+\mu \Rightarrow 3\lambda - \mu = -\frac{1}{2}$ **(2)**

Subtracting **(1)** from **(2)** gives:

$3\lambda - \mu - 2\lambda + \mu = -\frac{1}{2} - 0$

$\lambda = -\frac{1}{2}$

Substituting $\lambda = -\frac{1}{2}$ into **(1)** gives:

$2\left(-\frac{1}{2}\right) - \mu = 0$

$\mu = -1$

Checking that $\lambda = -\frac{1}{2}$ and $\mu = -1$ satisfy $5+\lambda = \frac{5}{2} - 2\mu$ gives:

$\text{LHS} = 5 + \left(-\frac{1}{2}\right)$

$\text{RHS} = \frac{5}{2} - 2(-1)$

$\frac{9}{2} = \frac{9}{2}$

So the lines do intersect.

2 l_1 has equation $\mathbf{r} = \begin{pmatrix}-6\\0\\11\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\1\end{pmatrix}$

l_2 has equation $\mathbf{r} = \begin{pmatrix}2\\-2\\9\end{pmatrix} + \mu\begin{pmatrix}2\\1\\-3\end{pmatrix}$

$\begin{pmatrix}-6+\lambda\\-\lambda\\11+\lambda\end{pmatrix} = \begin{pmatrix}2+2\mu\\-2+\mu\\9-3\mu\end{pmatrix}$

$-6+\lambda = 2+2\mu \Rightarrow \lambda - 2\mu = 8$ **(1)**

$-\lambda = -2+\mu \Rightarrow \lambda + \mu = 2$ **(2)**

Adding **(1)** and 2 × **(2)** gives:

$\lambda - 2\mu + 2\lambda + 2\mu = 8 + 4$

$3\lambda = 12$

$\lambda = 4$

Substituting $\lambda = 4$ into **(1)** gives:

$(4) - 2\mu = 8$

$\mu = -2$

Checking that $\lambda = 4$ and $\mu = -2$ satisfy $11+\lambda = 9-3\mu$ gives:

$\text{LHS} = 11 + (4)$

$\text{RHS} = 9 - 3(-2)$

$15 = 15$

So the lines do intersect.

Substituting $\lambda = 4$ into $\begin{pmatrix}-6\\0\\11\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\1\end{pmatrix}$ gives:

$\begin{pmatrix}-6\\0\\11\end{pmatrix} + 4\begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}-2\\-4\\15\end{pmatrix}$

The point where the lines meet is (−2, −4, 15)

3 l_1 has equation $\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$

l_2 has equation $\mathbf{r} = \begin{pmatrix} 5 \\ 4 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$

$\begin{pmatrix} 3+2\lambda \\ 1+2\lambda \\ -2+3\lambda \end{pmatrix} = \begin{pmatrix} 5+2\mu \\ 4+\mu \\ -\mu \end{pmatrix}$

$3+2\lambda = 5+2\mu \Rightarrow \lambda - \mu = 1$ **(1)**

$1+2\lambda = 4+\mu \Rightarrow 2\lambda - \mu = 3$ **(2)**

Subtracting **(1)** from **(2)** gives:

$2\lambda - \mu - \lambda + \mu = 3-1$

$\lambda = 2$

Substituting $\lambda = 2$ into **(1)** gives:

$(2) - \mu = 1$

$\mu = 1$

Checking that $\lambda = 2$ and $\mu = 1$ satisfy $-2+3\lambda = -\mu$ gives:

$\text{LHS} = -2+3(2)$

$\text{RHS} = -(1)$

$4 \neq -1$

So the lines do not intersect.

4 a l_1 has equation $\mathbf{r} = \begin{pmatrix} 5 \\ 4 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$

l_2 has equation $\mathbf{r} = \begin{pmatrix} 0 \\ 11 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ p \\ p \end{pmatrix}$

If the lines are perpendicular then:

$\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ p \\ p \end{pmatrix} = 0$

$-3 - p + 2p = 0$

$p = 3$

4 b $\begin{pmatrix} 5+3\lambda \\ 4-\lambda \\ -1+2\lambda \end{pmatrix} = \begin{pmatrix} -\mu \\ 11+3\mu \\ 3+3\mu \end{pmatrix}$

$5+3\lambda = -\mu \Rightarrow 3\lambda + \mu = -5$ **(1)**

$4-\lambda = 11+3\mu \Rightarrow \lambda + 3\mu = -7$ **(2)**

Subtracting 3 × **(2)** from **(1)** gives:

$3\lambda + \mu - 3\lambda - 9\mu = -5 + 21$

$-8\mu = 16$

$\mu = -2$

Substituting $\mu = -2$ into **(1)** gives:

$3\lambda + (-2) = -5$

$\lambda = -1$

Checking that $\lambda = -1$ and $\mu = -2$ satisfy $-1+2\lambda = 3+3\mu$ gives:

$\text{LHS} = -1+2(-1)$

$\text{RHS} = 3+3(-2)$

$-3 = -3$

So the lines do intersect.

Substituting $\lambda = -1$ into $\begin{pmatrix} 5 \\ 4 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ gives:

$\begin{pmatrix} 5 \\ 4 \\ -1 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix}$

The point where the lines meet is (2, 5, −3)

5 l_1 has equation $\mathbf{r} = \begin{pmatrix}5\\2\\1\end{pmatrix} + \lambda\begin{pmatrix}-1\\1\\2\end{pmatrix}$

l_2 has equation $\mathbf{r} = \begin{pmatrix}4\\1\\1\end{pmatrix} + \mu\begin{pmatrix}1\\0\\-1\end{pmatrix}$

$\begin{pmatrix}5-\lambda\\2+\lambda\\1+2\lambda\end{pmatrix} = \begin{pmatrix}4+\mu\\1\\1-\mu\end{pmatrix}$

$2+\lambda = 1 \Rightarrow \lambda = -1$

$5-\lambda = 4+\mu \Rightarrow \mu = 2$

Checking that $\lambda = -1$ and $\mu = 2$ satisfy $1+2\lambda = 1-\mu$ gives:

$1+2(-1) = 1-(2)$

$-1 = -1$

So the lines do intersect.

Substituting $\lambda = -1$ into $\begin{pmatrix}5\\2\\1\end{pmatrix} + \lambda\begin{pmatrix}-1\\1\\2\end{pmatrix}$ gives:

$\begin{pmatrix}5\\2\\1\end{pmatrix} - \begin{pmatrix}-1\\1\\2\end{pmatrix} = \begin{pmatrix}6\\1\\-1\end{pmatrix}$

The point where the lines meet is (6, 1, −1)

6 a l_1 has equation $\mathbf{r} = \begin{pmatrix}8\\2\\-12\end{pmatrix} + \lambda\begin{pmatrix}-1\\3\\2\end{pmatrix}$

l_2 has equation $\mathbf{r} = \begin{pmatrix}-4\\10\\p\end{pmatrix} + \mu\begin{pmatrix}q\\2\\-1\end{pmatrix}$

If the lines are perpendicular then:

$\begin{pmatrix}-\lambda\\3\lambda\\2\lambda\end{pmatrix} \cdot \begin{pmatrix}\mu q\\2\mu\\-\mu\end{pmatrix} = 0$

$-\lambda\mu q + 6\lambda\mu - 2\lambda\mu = 0$

$4\lambda\mu = \lambda\mu q$

$q = 4$ as required

6 b Since the lines intersect:

$\begin{pmatrix}8-\lambda\\2+3\lambda\\-12+2\lambda\end{pmatrix} = \begin{pmatrix}-4+4\mu\\10+2\mu\\p-\mu\end{pmatrix}$

$8-\lambda = -4+4\mu \Rightarrow \lambda + 4\mu = 12$ **(1)**

$2+3\lambda = 10+2\mu \Rightarrow 3\lambda - 2\mu = 8$ **(2)**

Adding 2 × **(2)** to **(1)** gives:

$\lambda + 4\mu + 6\lambda - 4\mu = 12 + 16$

$7\lambda = 28$

$\lambda = 4$

Substituting $\lambda = 4$ into **(1)** gives:

$4 + 4\mu = 12$

$\mu = 2$

Substituting $\lambda = 4$ and $\mu = 2$ into $-12 + 2\lambda = p - \mu$ gives:

$-12 + 2(4) = p - 2$

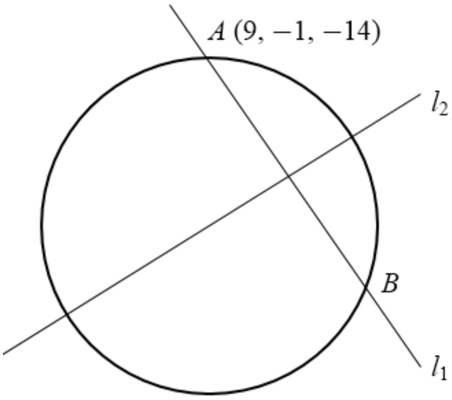
$p = -2$

c Substituting $\lambda = 4$ into $\begin{pmatrix}8\\2\\-12\end{pmatrix} + \lambda\begin{pmatrix}-1\\3\\2\end{pmatrix}$ gives:

$\begin{pmatrix}8\\2\\-12\end{pmatrix} + 4\begin{pmatrix}-1\\3\\2\end{pmatrix} = \begin{pmatrix}4\\14\\-4\end{pmatrix}$

Therefore, the point where the lines meet is (4, 14, −4)

6 d



The position vector from A to the point where the lines meet is:

$$\begin{pmatrix} 4 \\ 14 \\ -4 \end{pmatrix} - \begin{pmatrix} 9 \\ -1 \\ -14 \end{pmatrix} = \begin{pmatrix} -5 \\ 15 \\ 10 \end{pmatrix}$$

Therefore, the position vector from the point where the lines meet to B is:

$$\begin{pmatrix} 4 \\ 14 \\ -4 \end{pmatrix} + \begin{pmatrix} -5 \\ 15 \\ 10 \end{pmatrix} = \begin{pmatrix} -1 \\ 29 \\ 6 \end{pmatrix}$$

So B has coordinates $(-1, 29, 6)$