Exercise 7I

1 a $\mathbf{a} = 6\mathbf{i} + 5\mathbf{j} - \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - 3\mathbf{j} - \mathbf{k}$

An equation of the line is:

$$\mathbf{r} = \begin{pmatrix} 6 \\ 5 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}$$

b $\mathbf{a} = 2\mathbf{i} + 5\mathbf{j}$, $\mathbf{b} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

An equation of the line is:

$$\mathbf{r} = \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

c $\mathbf{a} = -7\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

An equation of the line is:

$$\mathbf{r} = \begin{pmatrix} -7 \\ 6 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

d $\mathbf{a} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$

An equation of the line is:

$$\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$$

e $\mathbf{a} = \begin{pmatrix} 6 \\ -11 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 0 \\ 5 \\ -2 \end{pmatrix}$

An equation of the line is:

$$\mathbf{r} = \begin{pmatrix} 6 \\ -11 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 5 \\ -2 \end{pmatrix}$$

2 a i $\mathbf{p} = 3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$, $\mathbf{q} = 5\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$
$$= \begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix} - \begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix}$$
$$= \begin{pmatrix} 2 \\ 7 \\ -3 \end{pmatrix}$$

ii An equation of the line is:

$$\mathbf{r} = \begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 7 \\ -3 \end{pmatrix}$$

b i $\mathbf{p} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$, $\mathbf{q} = 4\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$
$$= \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$$
$$= \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$$

ii An equation of the line is:

$$\mathbf{r} = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$$

c i $\mathbf{p} = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$, $\mathbf{q} = -2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$
$$= \begin{pmatrix} -2 \\ -3 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$$
$$= \begin{pmatrix} -3 \\ -1 \\ -2 \end{pmatrix}$$

2 c ii An equation of the line is:

$$\mathbf{r}=\begin{pmatrix}1\\-2\\4\end{pmatrix}+\lambda\begin{pmatrix}-3\\-1\\-2\end{pmatrix}$$

d i $\mathbf{p}=\begin{pmatrix}3\\-1\\4\end{pmatrix}$, $\mathbf{q}=\begin{pmatrix}-2\\3\\1\end{pmatrix}$

$$\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}$$
$$=\begin{pmatrix}-2\\3\\1\end{pmatrix}-\begin{pmatrix}3\\-1\\4\end{pmatrix}$$
$$=\begin{pmatrix}-5\\4\\-3\end{pmatrix}$$

ii An equation of the line is:

$$\mathbf{r}=\begin{pmatrix}3\\-1\\4\end{pmatrix}+\lambda\begin{pmatrix}-5\\4\\-3\end{pmatrix}$$

e i $\mathbf{p}=\begin{pmatrix}4\\-2\\3\end{pmatrix}$, $\mathbf{q}=\begin{pmatrix}-2\\2\\4\end{pmatrix}$

$$\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}$$
$$=\begin{pmatrix}-2\\2\\4\end{pmatrix}-\begin{pmatrix}4\\-2\\3\end{pmatrix}$$
$$=\begin{pmatrix}-6\\4\\1\end{pmatrix}$$

ii An equation of the line is:

$$\mathbf{r}=\begin{pmatrix}4\\-2\\3\end{pmatrix}+\lambda\begin{pmatrix}-6\\4\\1\end{pmatrix}$$

3 $\mathbf{r}=\begin{pmatrix}4\\-3\\8\end{pmatrix}+\lambda\begin{pmatrix}0\\0\\1\end{pmatrix}$

4 a $\mathbf{r}=\begin{pmatrix}2\\-3\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\-4\\-9\end{pmatrix}$

Since $(1, p, q)$ lies on the line:

$$\begin{pmatrix}2\\-3\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\-4\\-9\end{pmatrix}=\begin{pmatrix}1\\p\\q\end{pmatrix}$$

$2+\lambda=1\Rightarrow\lambda=-1$

$-3-4\lambda=p\Rightarrow p=-3-4(-1)\Rightarrow p=1$

$1-9\lambda=q\Rightarrow q=1-9(-1)\Rightarrow q=10$

b $\mathbf{r}=\begin{pmatrix}-4\\6\\-1\end{pmatrix}+\lambda\begin{pmatrix}2\\-5\\-8\end{pmatrix}$

Since $(1, p, q)$ lies on the line:

$$\begin{pmatrix}-4\\6\\-1\end{pmatrix}+\lambda\begin{pmatrix}2\\-5\\-8\end{pmatrix}=\begin{pmatrix}1\\p\\q\end{pmatrix}$$

$-4+2\lambda=1\Rightarrow 2\lambda=5\Rightarrow\lambda=\frac{5}{2}$

$6-5\lambda=p\Rightarrow p=6-5\left(\frac{5}{2}\right)\Rightarrow p=-\frac{13}{2}$

$-1-8\lambda=q\Rightarrow q=-1-8\left(\frac{5}{2}\right)\Rightarrow q=-21$

c $\mathbf{r}=\begin{pmatrix}16\\-9\\-10\end{pmatrix}+\lambda\begin{pmatrix}3\\2\\1\end{pmatrix}$

Since $(1, p, q)$ lies on the line:

$$\begin{pmatrix}16\\-9\\-10\end{pmatrix}+\lambda\begin{pmatrix}3\\2\\1\end{pmatrix}=\begin{pmatrix}1\\p\\q\end{pmatrix}$$

$16+3\lambda=1\Rightarrow 3\lambda=-15\Rightarrow\lambda=-5$

$-9+2\lambda=p\Rightarrow p=-9+2(-5)\Rightarrow p=-19$

$-10+\lambda=q\Rightarrow q=-10+(-5)\Rightarrow q=-15$

5 $\mathbf{r} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$

The direction of l_1 is $\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$

B is the point (3, 7, −5)

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= \begin{pmatrix} 3 \\ 7 \\ -5 \end{pmatrix} - \begin{pmatrix} 5 \\ 4 \\ -1 \end{pmatrix}$

$= \begin{pmatrix} -2 \\ 3 \\ -4 \end{pmatrix}$

The direction of l_2 is $\begin{pmatrix} -2 \\ 3 \\ -4 \end{pmatrix} = -1 \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$

Therefore l_1 and l_2 are parallel.

6 $A(-3,-4,5)$, $B(3,-1,2)$, $A(9,2,-1)$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} -3 \\ -4 \\ 5 \end{pmatrix}$

$= \begin{pmatrix} 6 \\ 3 \\ -3 \end{pmatrix}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$= \begin{pmatrix} 9 \\ 2 \\ -1 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} 6 \\ 3 \\ -3 \end{pmatrix}$

Since the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are in the same direction and they have a point in common they are collinear.

7 $\begin{pmatrix} 3 \\ -1 \\ 8 \end{pmatrix} - \begin{pmatrix} 1 \\ 7 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -8 \\ 10 \end{pmatrix}$

$\begin{pmatrix} 10 \\ 4 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 7 \\ -2 \end{pmatrix} = \begin{pmatrix} 9 \\ -3 \\ 2 \end{pmatrix}$

Therefore not collinear.

8 $P(2,0,4)$, $Q(a,5,1)$, $R(3,10,b)$

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$= \begin{pmatrix} a \\ 5 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}$

$= \begin{pmatrix} a-2 \\ 5 \\ -3 \end{pmatrix}$

$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP}$

$= \begin{pmatrix} 3 \\ 10 \\ b \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ 10 \\ b-4 \end{pmatrix}$

Since the points are collinear:

$k \begin{pmatrix} a-2 \\ 5 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 10 \\ b-4 \end{pmatrix}$ therefore $k = 2$

$2(a-2) = 1$

$2a - 4 = 1$

$a = \frac{5}{2}$

$2(-3) = b - 4$

$-6 = b - 4$

$b = -2$

9 $\mathbf{r} = (8\mathbf{i} - 5\mathbf{j} + 4\mathbf{k}) + \lambda(3\mathbf{i} + \mathbf{j} - 6\mathbf{k})$

A lies on l_1 where $\lambda = -2$
Therefore A is the point:

$$\begin{pmatrix} 8 \\ -5 \\ 4 \end{pmatrix} - 2\begin{pmatrix} 3 \\ 1 \\ -6 \end{pmatrix} = \begin{pmatrix} 2 \\ -7 \\ 16 \end{pmatrix}$$

$\mathbf{r} = (10\mathbf{i} + 3\mathbf{j} - 9\mathbf{k}) + \lambda(2\mathbf{i} - 4\mathbf{j} + \mathbf{k})$

l_2 passes through A, therefore:

$$\mathbf{r} = \begin{pmatrix} 2 \\ -7 \\ 16 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix}$$

10 a The line L is

$\mathbf{r} = (10\mathbf{i} + 8\mathbf{j} - 12\mathbf{k}) + \lambda(\mathbf{i} - \mathbf{j} + b\mathbf{k})$

The point A is $(4, a, 0)$

$$\begin{pmatrix} 10 \\ 8 \\ -12 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -1 \\ b \end{pmatrix} = \begin{pmatrix} 4 \\ a \\ 0 \end{pmatrix}$$

$10 + \lambda = 4$

$\lambda = -6$

$8 - \lambda = a$

$8 - (-6) = a$

$a = 14$

$-12 + \lambda b = 0$

$-12 - 6b = 0$

$b = -2$

b X lies on L where $\lambda = -1$

$$\begin{pmatrix} 10 \\ 8 \\ -12 \end{pmatrix} - 1\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 9 \\ 9 \\ -10 \end{pmatrix}$$

X has coordinates $(9, 9, -10)$

11 The line l has equation

$$\mathbf{r} = \begin{pmatrix} 3 \\ -5 \\ 9 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

A is the point where $\lambda = 5$, therefore:

$$\begin{pmatrix} 3 \\ -5 \\ 9 \end{pmatrix} + 5\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 8 \\ 5 \\ -1 \end{pmatrix}$$

A has coordinates $(8, 5, -1)$
B is the point where $\lambda = 2$, therefore:

$$\begin{pmatrix} 3 \\ -5 \\ 9 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ 5 \end{pmatrix}$$

B has coordinates $(5, -1, 5)$
The length of AB is:

$$|AB| = \sqrt{(8-5)^2 + (5-(-1))^2 + (-1-5)^2}$$
$$= \sqrt{3^2 + 6^2 + (-6)^2}$$
$$= \sqrt{81}$$
$$= 9$$

12 The line l has equation

$$\mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

C is the point where $\lambda = 4$, therefore:

$$\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + 4\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 9 \\ 2 \\ -1 \end{pmatrix}$$

C has coordinates $(9, 2, -1)$
A is the point where $\lambda = 3$, therefore:

$$\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + 3\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ 0 \end{pmatrix}$$

A has coordinates $(7, 1, 0)$
The circle has centre C and since B lies on the diameter of the circle,
B has coordinates $(11, 3, -2)$

13 a The line l_1 has equation

$$\mathbf{r} = \begin{pmatrix} -4 \\ 6 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

A is the point where $\lambda = 2$, therefore:

$$\begin{pmatrix} -4 \\ 6 \\ 5 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ 7 \end{pmatrix}$$

A has position vector $\begin{pmatrix} -2 \\ 4 \\ 7 \end{pmatrix}$

B is the point where $\lambda = 5$, therefore:

$$\begin{pmatrix} -4 \\ 6 \\ 5 \end{pmatrix} + 5 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 10 \end{pmatrix}$$

B has position vector $\begin{pmatrix} 1 \\ 1 \\ 10 \end{pmatrix}$

b P has position vector $\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$

Therefore l_2 has equation:

$$\mathbf{r} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

13 c $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$$\overrightarrow{AB} = \begin{pmatrix} 1 \\ 1 \\ 10 \end{pmatrix} - \begin{pmatrix} -2 \\ 4 \\ 7 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix}$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} \lambda \\ 2-\lambda \\ 3+\lambda \end{pmatrix} - \begin{pmatrix} -2 \\ 4 \\ 7 \end{pmatrix} = \begin{pmatrix} \lambda+2 \\ -2-\lambda \\ -4+\lambda \end{pmatrix}$$

$\overrightarrow{AD}$ is of the same form.

Since $AB = AC = AD$,

$$\left| \begin{pmatrix} \lambda+2 \\ -2-\lambda \\ -4+\lambda \end{pmatrix} \right| = \left| \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} \right|$$

$\lambda^2 = 1$, so $\lambda = \pm 1$

C has position vector:

$$\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$$

D has position vector:

$$\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$$

The midpoint of CD is given by:

$$\left(\frac{1-1}{2}, \frac{1+3}{2}, \frac{4+2}{2} \right) = (0, 2, 3)$$

Therefore P is the midpoint of CD.

14 a Let P be the point (2, 3, 8) and Q be the point (22, 18, 8)
Let A be the point (a, b, c)
Let the R be the point (14, 1, 0) and S be the point (6, 17, 0)
$|RA| = |AS| = 12$

$$|RA| = \sqrt{(14-a)^2+(1-b)^2+(0-c)^2} = 12$$
$$196-28a+a^2+1-2b+b^2+c^2=144$$
$$a^2+b^2+c^2=28a+2b-53$$
$$|AS| = \sqrt{(a-6)^2+(b-17)^2+(c-0)^2} = 12$$
$$a^2-12a+36+b^2-34b+289+c^2=144$$
$$a^2+b^2+c^2=12a+34b-181$$

So:
$$28a+2b-53=12a+34b-181$$
$$16a-32b=-128 \Rightarrow a-2b=-8$$
$$\overrightarrow{PQ}=\begin{pmatrix}22\\18\\8\end{pmatrix}-\begin{pmatrix}2\\3\\8\end{pmatrix}=\begin{pmatrix}20\\15\\0\end{pmatrix}$$
$$\overrightarrow{PA}=\begin{pmatrix}a\\b\\c\end{pmatrix}-\begin{pmatrix}2\\3\\8\end{pmatrix}=\begin{pmatrix}a-2\\b-3\\c-8\end{pmatrix}$$
$$\overrightarrow{PQ}=k\overrightarrow{PA}$$
$$\begin{pmatrix}20\\15\\0\end{pmatrix}=k\begin{pmatrix}a-2\\b-3\\c-8\end{pmatrix}$$
$$\begin{pmatrix}20\\15\\0\end{pmatrix}=k\begin{pmatrix}a-2\\b-3\\c-8\end{pmatrix}$$

Substituting $a=2b-8$ gives:
$$\begin{pmatrix}20\\15\\0\end{pmatrix}=k\begin{pmatrix}2b-10\\b-3\\c-8\end{pmatrix}$$
$$k(2b-10)=20 \Rightarrow k=\frac{20}{2b-10}$$
$$k(b-3)=15 \Rightarrow k=\frac{15}{b-3}$$

Equating for k gives:
$$\frac{20}{2b-10}=\frac{15}{b-3}$$
$$20b-60=30b-150$$
$$10b=90$$
$$b=9$$

Substituting $b=9$ into $a=2b-8$ gives:
$$a=2(9)-8$$
$$=10$$
$$k(c-8)=0$$

Therefore $c = 8$
So the coordinates of A are (10, 9, 8)

b The tightrope will bow in the middle due to the acrobat's weight.