Chapter review 6

1 $x = 1 - t^2,\ y = t^3 + 1,\ -2 \le t \le 0$

$$\frac{dx}{dt} = -2t \Rightarrow dx = -2t\ dt$$

$$A = \int_{-2}^{0} y\ dx$$

$$= \int_{-2}^{0} (t^3 + 1)(-2t)\ dt$$

$$= -2\int_{-2}^{0} (t^4 + t)\ dt$$

$$= -2\left[\frac{1}{5}t^5 + \frac{1}{2}t^2\right]_{-2}^{0}$$

$$= -2\left[\left(\frac{1}{5}(0)^5 + \frac{1}{2}(0)^2\right) - \left(\frac{1}{5}(-2)^5 + \frac{1}{2}(-2)^2\right)\right]$$

$$= -2\left(0 - \left(-\frac{32}{5} + 2\right)\right)$$

$$= -\frac{44}{5}$$

2 $x = \ln(t+2),\ y = 4t,\ 3 \le t \le 13$

$$\frac{dx}{dt} = \frac{1}{t+2} \Rightarrow dx = \frac{1}{t+2}dt$$

$$A = \int_{3}^{13} y\ dx$$

$$= \int_{3}^{13} (4t)\frac{1}{t+2}\ dt$$

$$= 4\int_{3}^{13} \frac{t}{t+2}\ dt$$

Let $u = t + 2$

$$\frac{du}{dt} = 1 \Rightarrow du = dt$$

$$4\int_{3}^{13} \frac{t}{t+2}\ dt = 4\int_{5}^{15}\left(\frac{u-2}{u}\right) du$$

$$= 4\int_{5}^{15}\left(1 - \frac{2}{u}\right) du$$

$$= 4\left[u - 2\ln|u|\right]_{5}^{15}$$

$$= 4\left[(15 - 2\ln 15) - (5 - 2\ln 5)\right]$$

$$= 4(10 + 2\ln 5 - 2\ln 15)$$

$$= 40 + 8\ln\left(\frac{1}{3}\right)$$

$$= 40 - 8\ln 3$$

3 $V = \pi\int_0^3 y^2\ dx$

$$= \pi\int_0^3 x^4(9 - x^2)\ dx$$

$$= \pi\int_0^3 (9x^4 - x^6)\ dx$$

$$= \pi\left[\frac{9x^5}{5} - \frac{x^7}{7}\right]_0^3$$

$$= \pi\left(\frac{2187}{5} - \frac{2187}{7}\right)$$

$$= \frac{4374}{35}\pi$$

4 a $2y^2 - 6\sqrt{x} + 3 = 0$

$y = 0 \Rightarrow 3 = 6\sqrt{x} \Rightarrow x = \frac{1}{4}$

Curve cuts the x-axis when $x = \frac{1}{4}$

b $V = \pi\int_{0.25}^{4} y^2\ dx$

$$= \pi\int_{0.25}^{4}\left(3\sqrt{x} - \tfrac{3}{2}\right) dx$$

$$= 3\pi\int_{0.25}^{4}\left(\sqrt{x} - \tfrac{1}{2}\right) dx$$

$$= 3\pi\left[\tfrac{2}{3}x^{\frac{3}{2}} - \tfrac{1}{2}x\right]_{0.25}^{4}$$

$$= 3\pi\left(\left(\tfrac{16}{3} - 2\right) - \left(\tfrac{1}{12} - \tfrac{1}{8}\right)\right)$$

$$= 3\pi\left(\tfrac{10}{3} + \tfrac{1}{24}\right)$$

$$= 3\pi \times \tfrac{81}{24}$$

$$= \tfrac{81}{8}\pi$$

5 a $f(x) = x^2 + 4x + 4$

$y = f(x) \Rightarrow y = x^2 + 4x + 4$

$y = (x+2)^2$

$\sqrt{y} = x + 2$

$x = \sqrt{y} - 2$

$x^2 = (\sqrt{y} - 2)^2$

$x^2 = 4 - 4\sqrt{y} + y$

5 b $V = \pi\int_4^9 x^2\,dy$

$= \pi\int_4^9 \left(4 - 4\sqrt{y} + y\right) dy$

$= \pi\left[4y - \frac{8}{3}y^{\frac{3}{2}} + \frac{1}{2}y^2\right]_4^9$

$= \pi\left(\left(36 - 72 + \frac{81}{2}\right) - \left(16 - \frac{64}{3} + 8\right)\right)$

$= \pi\left(\frac{9}{2} - \frac{8}{3}\right)$

$= \frac{11}{6}\pi$

6 $V = \pi\int_0^1 y^2\,dx$

$= \pi\int_0^1 \left(x^2 + 3\right)^2 dx$

$= \pi\int_0^1 \left(x^4 + 6x^2 + 9\right) dx$

$= \pi\left[\frac{1}{5}x^5 + 2x^3 + 9x\right]_0^1$

$= \frac{56}{5}\pi$

7 a Substituting $x = 2$ and $y = 4.5$ into the equation of the line gives
$3 \times 2 + 4 \times 4.5 = 24$
Substituting $x = 2$ into the equation of the curve gives
$y = \frac{1}{4} \times 2 \times (2 + 1)^2 = 4.5$
So (2, 4.5) are the coordinates of the point of intersection A

7 b Volume V_1 is generated by the curve between O and A

$V_1 = \pi\int_0^2 y^2\,dx$

$= \frac{\pi}{16}\int_0^2 x^2(x+1)^4\,dx$

$= \frac{\pi}{16}\int_0^2 x^2\left(x^4 + 4x^3 + 6x^2 + 4x + 1\right) dx$

$= \frac{\pi}{16}\int_0^2 \left(x^6 + 4x^5 + 6x^4 + 4x^3 + x^2\right) dx$

$= \frac{\pi}{16}\left[\frac{1}{7}x^7 + \frac{2}{3}x^6 + \frac{6}{5}x^5 + x^4 + \frac{1}{3}x^3\right]_0^2$

$= \frac{\pi}{16}\left(\frac{128}{7} + \frac{128}{3} + \frac{192}{5} + 16 + \frac{8}{3}\right)$

$= \pi\left(\frac{8}{7} + \frac{8}{3} + \frac{12}{5} + 1 + \frac{1}{6}\right)$

$= \pi\left(\frac{240 + 560 + 504 + 210 + 35}{210}\right)$

$= \frac{1549}{210}\pi$

Volume V_2 generated by the line $3x + 4y = 24$ is a cone with $r = 4.5$ and $h = 6$

$V_2 = \frac{1}{3}\pi \times \left(\frac{9}{2}\right)^2 \times 6$

$= \frac{81}{2}\pi$

Total volume $= \frac{1549}{210}\pi + \frac{81}{2}\pi = \frac{5027}{105}\pi$

8 $x = t^{\frac{1}{2}},\ y = 2t^{\frac{1}{2}},\ 1 \le t \le 4$

$\frac{dx}{dt} = \frac{1}{2}t^{-\frac{1}{2}} \Rightarrow dx = \frac{1}{2}t^{-\frac{1}{2}}\ dt$

$V = \pi\int_1^4 y^2\ dx$

$= \pi\int_1^4 (4t)\frac{1}{2}t^{-\frac{1}{2}}\ dt$

$= 2\pi\int_1^4 t^{\frac{1}{2}} dt$

$= 2\pi\left[\frac{2}{3}t^{\frac{3}{2}}\right]_1^4$

$= \frac{4}{3}\pi\left((4)^{\frac{3}{2}} - (1)^{\frac{3}{2}}\right)$

$= \frac{28}{3}\pi$

9 $x = t^3,\ y = 3t,\ 0.4 \le t \le 0.5$

$\frac{dx}{dt} = 3t^2 \Rightarrow dx = 3t^2\ dt$

$V = \pi\int_{0.4}^{0.5} y^2\ dx$

$= \pi\int_{0.4}^{0.5} (9t^2)3t^2\ dt$

$= 27\pi\int_{0.4}^{0.5} t^4\ dt$

$= 27\pi\left[\frac{1}{5}t^5\right]_{0.4}^{0.5}$

$= \frac{27}{5}\pi\left(0.5^5 - 0.4^5\right)$

$= 0.3564...$

$= 0.356$ (3 s.f.)

10 $x = 3t^3,\ y = \frac{1}{3}t^{-\frac{3}{2}},\ 8 \le t \le 128$

$\frac{dx}{dt} = 9t^2 \Rightarrow dx = 9t^2\ dt$

$V = \pi\int_{0.4}^{0.5} y^2\ dx$

$= \pi\int_8^{128} \left(\frac{1}{9}t^{-3}\right)9t^2\ dt$

$= \pi\int_8^{128} \frac{1}{t}\ dt$

$= \pi\left[\ln|t|\right]_8^{128}$

$= \pi\left(\ln 128 - \ln 8\right)$

$= \pi \ln 16$

11 a $I = \int x\sqrt{4x-1}\ dx$

Let $u = 4x - 1 \Rightarrow \frac{du}{dx} = 4$

$I = \int \frac{u+1}{16}\sqrt{u}\ du$

$= \frac{2u^{\frac{5}{2}}}{80} + \frac{2u^{\frac{3}{2}}}{48} + c$

$= \frac{(4x-1)^{\frac{5}{2}}}{40} + \frac{(4x-1)^{\frac{3}{2}}}{24} + c$

b $I = \int x \ln x\ dx$

Let $u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = x \Rightarrow v = \frac{x^2}{2}$

$I = \frac{x^2}{2}\ln x - \int \frac{x^2}{2} \times \frac{1}{x}\ dx$

$= \frac{x^2}{2}\ln x - \frac{x^2}{4} + c$

c $I = \int \frac{4\sin x \cos x}{4 - 8\sin^2 x}\ dx$

$I = \int \frac{2\sin 2x}{4\left(1 - 2\sin^2 x\right)}\ dx$

$I = \int \frac{2\sin 2x}{4\cos 2x}\ dx$

$= -\tfrac{1}{4}\ln|\cos 2x| + c$

12 a $I=\int_0^{\frac{\pi}{4}} x\sec^2 x \, \mathrm{d}x$

Let $u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x}=1$

$\frac{\mathrm{d}v}{\mathrm{d}x}=\sec^2 x \Rightarrow v=\tan x$

$I=\left[x\tan x\right]_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}\tan x \, \mathrm{d}x$

$=\frac{\pi}{4}+\left[\ln|\cos x|\right]_0^{\frac{\pi}{4}}=\frac{\pi}{4}+\ln\frac{1}{\sqrt{2}}$

$=\frac{\pi}{4}-\frac{1}{2}\ln 2$

b $I=\int_1^4 \frac{4}{16x^2+8x-3} \, \mathrm{d}x$

$\frac{4}{16x^2+8x-3}=\frac{4}{(4x+3)(4x-1)}$

$\frac{4}{(4x+3)(4x-1)}=\frac{A}{4x+3}+\frac{B}{4x-1}$

$4=A(4x-1)+B(4x+3)$

$x=\frac{1}{4}\Rightarrow 4=4B\Rightarrow B=1$

$x=-\frac{3}{4}\Rightarrow 4=-4A\Rightarrow A=-1$

$I=\int_1^4 \frac{1}{4x-1}-\frac{1}{4x+3} \, \mathrm{d}x$

$=\frac{1}{4}\left[\ln|4x-1|-\ln|4x+3|\right]_1^4$

$=\frac{1}{4}\left(\ln 15-\ln 19-\ln 3+\ln 7\right)$

$=\frac{1}{4}\ln\frac{105}{57}$

$=\frac{1}{4}\ln\frac{35}{19}$

c $I=\int_0^{\ln 2}\frac{1}{1+\mathrm{e}^x} \, \mathrm{d}x$

Let $u=1+\mathrm{e}^x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x}=\mathrm{e}^x=u-1$

$I=\int_2^3 \frac{1}{(u-1)u} \, \mathrm{d}u$

$I=\int_2^3\left(\frac{1}{(u-1)}-\frac{1}{u}\right) \mathrm{d}u$

$=\left[\ln|u-1|-\ln|u|\right]_2^3$

$=\ln 2-\ln 3-\ln 1+\ln 2$

$=\ln\frac{4}{3}$

13 a $I=\int_1^{\mathrm{e}}\frac{1}{x^2}\ln x \, \mathrm{d}x$

$u=\ln x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{x}$

$\frac{\mathrm{d}v}{\mathrm{d}x}=\frac{1}{x^2}\Rightarrow v=-\frac{1}{x}$

$\therefore I=\left[-\frac{1}{x}\ln x\right]_1^{\mathrm{e}}-\int_1^{\mathrm{e}}\left(-\frac{1}{x^2}\right)\mathrm{d}x$

$=\left(-\frac{1}{\mathrm{e}}\right)-(0)+\left[-\frac{1}{x}\right]_1^{\mathrm{e}}$

$=-\frac{1}{\mathrm{e}}+\left(-\frac{1}{\mathrm{e}}\right)-(-1)$

$=1-\frac{2}{\mathrm{e}}$

b $\frac{1}{(x+1)(2x-1)}\equiv\frac{A}{x+1}+\frac{B}{2x-1}$

$\Rightarrow 1\equiv A(2x-1)+B(x+1)$

$x-\frac{1}{2}\Rightarrow 1=\frac{3}{2}B\Rightarrow B=\frac{2}{3}$

$x=-1\Rightarrow 1=-3A\Rightarrow A=-\frac{1}{3}$

$\therefore \int_1^p\frac{1}{(x+1)(2x-1)}\mathrm{d}x=\int_1^p\left(\frac{\frac{2}{3}}{2x-1}+\frac{-\frac{1}{3}}{x+1}\right)\mathrm{d}x$

$=\left[\frac{1}{3}\ln|2x-1|-\frac{1}{3}\ln|x+1|\right]_1^p$

$=\left[\frac{1}{3}\ln\left|\frac{2x-1}{x+1}\right|\right]_1^p$

$=\frac{1}{3}\ln\left(\frac{2p-1}{p+1}\right)-\left(\frac{1}{3}\ln\frac{1}{2}\right)$

$=\frac{1}{3}\ln\left(\frac{4p-2}{p+1}\right)$

14 a $t^2 = x+1 \Rightarrow 2t\,dt = dx$

$\therefore I = \int \frac{x}{\sqrt{x+1}}dx$

$= \int \frac{t^2-1}{t} \times 2t\,dt$

$= \int (2t^2 - 2)dt$

$= \frac{2}{3}t^3 - 2t + c$

$= \frac{2}{3}(x+1)^{\frac{3}{2}} - 2\sqrt{x+1} + c$

$= \frac{2}{3}\sqrt{x+1}(x-2) + c$

b $\int_0^3 \frac{x}{\sqrt{x+1}}dx = \left[\frac{2}{3}(x-2)\sqrt{x+1}\right]_0^3$

$= \left(\frac{2}{3}\times 2\right) - \left(-\frac{4}{3}\right) = \frac{8}{3}$

15 a $I = \int x\sin 8x\ dx$

Let $u = x \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = \sin 8x \Rightarrow v = -\frac{1}{8}\cos 8x$

$I = -\frac{1}{8}x\cos 8x + \frac{1}{8}\int \cos 8x\ dx$

$= -\frac{1}{8}x\cos 8x + \frac{1}{64}\sin 8x + c$

b $I = \int x^2 \cos 8x\ dx$

Let $u = x^2 \Rightarrow \frac{du}{dx} = 2x$

$\frac{dv}{dx} = \cos 8x \Rightarrow v = \frac{1}{8}\sin 8x$

$I = \frac{1}{8}x^2\sin 8x - \frac{2}{8}\int x\sin 8x\ dx$

$= \frac{1}{8}x^2\sin 8x - \frac{2}{8}\left(-\frac{1}{8}x\cos 8x + \frac{1}{64}\sin 8x\right) + c$

$= \frac{1}{8}x^2\sin 8x + \frac{1}{32}x\cos 8x - \frac{1}{256}\sin 8x + c$

16 a $f(x) \equiv \frac{5x^2-8x+1}{2x(x-1)^2} \equiv \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$

$\Rightarrow 5x^2 - 8x + 1 \equiv 2A(x-1)^2$
$+ 2Bx(x-1) + 2Cx$

$x = 0 \Rightarrow 1 = 2A \Rightarrow A = \frac{1}{2}$

$x = 1 \Rightarrow -2 = 2C \Rightarrow C = -1$

Coefficients of x^2: $5 = 2A + 2B \Rightarrow B = 2$

16 b $\int f(x)dx = \int\left(\frac{\frac{1}{2}}{x} + \frac{2}{x-1} - \frac{1}{(x-1)^2}\right)dx$

$= \frac{1}{2}\ln|x| + 2\ln|x-1| + \frac{1}{x-1} + c$

c $\int_4^9 f(x)dx = \left[\frac{1}{2}\ln|x| + 2\ln|x-1| + \frac{1}{x-1}\right]_4^9$

$= \left[\ln\left|\sqrt{x(x-1)^2}\right| + \frac{1}{x-1}\right]_4^9$

$= \left[\ln(3\times 64) + \frac{1}{8}\right] - \left[\ln(2\times 9) + \frac{1}{3}\right]$

$= \ln\left(\frac{3\times 64}{2\times 9}\right) + \frac{1}{8} - \frac{1}{3}$

$= \ln\frac{32}{3} - \frac{5}{24}$

17 a $I = \int x^2 \ln 2x\ dx$

Let $u = \ln 2x \Rightarrow \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = x^2 \Rightarrow v = \frac{x^3}{3}$

$I = \frac{1}{3}x^3\ln 2x - \int \frac{x^2}{3}\ dx$

$= \frac{1}{3}x^3\ln 2x - \frac{1}{9}x^3 + c$

b $\int_{\frac{1}{2}}^3 x^2\ln 2x\ dx = \left[\frac{1}{3}x^3\ln 2x - \frac{1}{9}x^3\right]_{\frac{1}{2}}^3$

$= 9\ln 6 - 3 - 0 + \frac{1}{72}$

$= 9\ln 6 - \frac{215}{9}$

18 a $I = \int xe^{-x}dx$

$u = x \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = e^{-x} \Rightarrow v = -e^{-x}$

$\therefore I = -xe^{-x} - \int(-e^{-x})dx$

i.e. $I = -xe^{-x} - e^{-x} + c$

18 b $e^x \frac{dy}{dx} = \frac{x}{\sin 2y}$

$\Rightarrow \int \sin 2y \, dy = \int xe^{-x} dx$

$\Rightarrow -\frac{1}{2}\cos 2y = -xe^{-x} - e^{-x} + c$

$x = 0, y = \frac{\pi}{4} \Rightarrow 0 = 0 - 1 + c \Rightarrow c = 1$

$\therefore \frac{1}{2}\cos 2y = xe^{-x} + e^{-x} - 1$

or $\cos 2y = 2(xe^{-x} + e^{-x} - 1)$

19 a $I = \int x \sin 2x \, dx$

$u = x \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = \sin 2x \Rightarrow v = -\frac{1}{2}\cos 2x$

$\therefore I = -\frac{1}{2}x\cos 2x - \int -\frac{1}{2}\cos 2x \, dx$

$= -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + c$

b $\frac{dy}{dx} = x \sin 2x \cos^2 y$

$\Rightarrow \int \sec^2 y \, dy = \int x \sin 2x \, dx$

$\Rightarrow \tan y = -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + c$

$y = 0,\ x = \frac{\pi}{4} \Rightarrow 0 = 0 + \frac{1}{4} + c \Rightarrow c = -\frac{1}{4}$

$\therefore \tan y = -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x - \frac{1}{4}$

20 a $\frac{dy}{dx} = xy^2$

$\Rightarrow \int \frac{1}{y^2} dy = \int x \, dx$

$\Rightarrow -\frac{1}{y} = \frac{x^2}{2} + c$

or $y = \frac{-2}{x^2 + k}$ $\quad (k = 2c)$

b $y = 1,\ x = 1 \Rightarrow 1 = \frac{-2}{1+k} \Rightarrow k = -3$

$\therefore y = \frac{2}{3 - x^2}$

for $x^2 \neq 3$ and $y > 0$, i.e. $-\sqrt{3} < x < \sqrt{3}$

20 c When $x = 1, y = 1,$ $\frac{dy}{dx}$ is 1

d Equation of tangent is:

$y - 1 = 1(x - 1)$

$y = x$

This meets the curve again when:

$x = \frac{2}{3 - x^2}$

$3x - x^3 = 2$

$x^3 - 3x + 2 = 0$

$(x-1)(x-1)(x+2) = 0$

Other point is when $x = -2, y = -2$

i.e. $(-2, -2)$

21 a $I = \int \frac{4x}{(1+2x)^2} dx$

$u = 1 + 2x$

$\Rightarrow \frac{du}{2} = dx$ and $4x = 2(u - 1)$

$\therefore I = \int \frac{2(u-1)}{u^2} \times \frac{du}{2}$

$= \int \left(\frac{1}{u} - u^{-2}\right) du$

$= \ln|u| + \frac{1}{u} + c$

$= \ln|1 + 2x| + \frac{1}{1+2x} + c$

21 b $(1+2x)^2 \frac{dy}{dx} = \frac{x}{\sin^2 y}$

$\Rightarrow \int \sin^2 y \, dy = \int \frac{x}{(1+2x)} dx$

$\Rightarrow \int 4\sin^2 y \, dy = \int \frac{4x}{(1+2x)^2} dx$

$\Rightarrow \int (2-2\cos 2y) \, dy = I$

$\Rightarrow 2y - \sin 2y = \ln|1+2x| + \frac{1}{1+2x} + c$

$x = 0, y = \frac{\pi}{4} \Rightarrow \frac{\pi}{2} - 1 = \ln 1 + 1 + c$

$\Rightarrow c = \frac{\pi}{2} - 2$

$\therefore 2y - \sin 2y = \ln|1+2x| + \frac{1}{1+2x} + \frac{\pi}{2} - 2$

22 a $\int xe^{2x} \, dx$

$u = x \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = e^{2x} \Rightarrow v = \frac{1}{2}e^{2x}$

$\therefore \int xe^{2x} \, dx = \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x} \, dx$

$= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c$

$A_1 = -\left[\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}\right]_{-\frac{1}{2}}^{0}$

$= -\left(\left(0 - \frac{1}{4}\right) - \left(-\frac{1}{4}e^{-1} - \frac{1}{4}e^{-1}\right)\right)$

$= \frac{1}{4}(1 - 2e^{-1})$

$A_2 = \left[\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}\right]_{0}^{\frac{1}{2}}$

$= \left(\frac{1}{4}e^{1} - \frac{1}{4}e^{1}\right) - \left(0 - \frac{1}{4}\right)$

$= \frac{1}{4}$

b $\frac{A_1}{A_2} = \frac{\frac{1}{4}\left(1-2e^{-1}\right)}{\frac{1}{4}} = 1 - 2e^{-1} = \frac{e-2}{e}$

$\therefore A_1 : A_2 = (e-2) : e$

23 a $I = \int x^2 e^{-x} \, dx$

Let $u = x^2 \Rightarrow \frac{du}{dx} = 2x$

$\frac{dv}{dx} = e^{-x} \Rightarrow v = -e^{-x}$

$I = -x^2 e^{-x} + 2\int xe^{-x} \, dx$

Again, let $u = x \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = e^{-x} \Rightarrow v = -e^{-x}$

$I = -x^2e^{-x} - 2xe^{-x} + 2\int e^{-x} \, dx$

$= -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + c$

$= -e^{-x}\left(x^2 + 2x + 2\right) + c$

b $\frac{dy}{dx} = x^2 e^{3y-x}$

$\frac{dy}{dx} = x^2 e^{3y} e^{-x}$

$\int e^{-3y} \, dx = \int x^2 e^{-x} \, dx$

$-\frac{1}{3}e^{-3y} = -e^{-x}\left(x^2 + 2x + 2\right) + c$

$x = 0, y = 0 \Rightarrow -\frac{1}{3} = -2 + c \Rightarrow c = \frac{5}{3}$

$e^{-3y} = 3e^{-x}\left(x^2 + 2x + 2\right) - 5$

$3y = -\ln\left(3e^{-x}\left(x^2 + 2x + 2\right) - 5\right)$

$y = -\frac{1}{3}\ln\left(3e^{-x}\left(x^2 + 2x + 2\right) - 5\right)$

24 a $\frac{x^2}{x^2 - 1} \equiv A + \frac{B}{x-1} + \frac{C}{x+1}$

$\Rightarrow x^2 \equiv A(x-1)(x+1) + B(x+1) + C(x-1)$

$x = 1 \Rightarrow 1 = 2B \Rightarrow B = \frac{1}{2}$

$x = -1 \Rightarrow 1 = -2C \Rightarrow C = -\frac{1}{2}$

Coefficients of x^2: $1 = A \Rightarrow A = 1$

24 b $\frac{dx}{dt} = 2\frac{(x^2-1)}{x^2}$

$\Rightarrow \int \frac{x^2}{x^2-1}dx = \int 2\,dt$

$\Rightarrow \int\left(1+\frac{\left(\frac{1}{2}\right)}{x-1}-\frac{\left(\frac{1}{2}\right)}{x+1}\right)dx = 2t$

$\Rightarrow x+\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| = 2t+c$

$x=2, t=1 \Rightarrow 2+\frac{1}{2}\ln\frac{1}{3} = 2+c \Rightarrow c = \frac{1}{2}\ln\frac{1}{3}$

$\therefore x+\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| = 2t+\frac{1}{2}\ln\frac{1}{3}$

25 a $\frac{dv}{dt} = -kV$

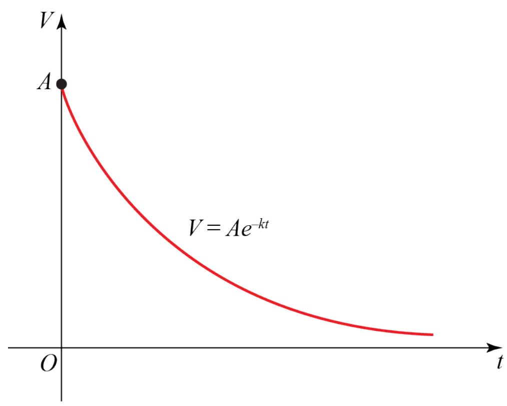
$\Rightarrow \int\frac{1}{V}dV = \int -k\,dt$

$\Rightarrow \ln|V| = -kt + C$

$\Rightarrow V = A_1e^{-kt}$

$t=0, V=A \Rightarrow V = Ae^{-kt} \qquad (A_1 = A)$

b



c $t=T, V=\frac{1}{2}A \Rightarrow \frac{1}{2}A = Ae^{-kT}$

$\Rightarrow -\ln 2 = -kT$

$\Rightarrow kT = \ln 2$

26 a $\frac{dy}{dx} = \frac{x}{k-y}$

$\int(k-y)\,dy = \int x\,dx$

$-\frac{(k-y)^2}{2}+c = \frac{x^2}{2}$

$x^2+(y-k)^2 = c$

b Concentric circles with centre (0, 2)

27 a $u = 1+2x^2 \Rightarrow du = 4x\,dx \Rightarrow x\,dx = \frac{du}{4}$

So

$\int x(1+2x^2)^5dx = \int\frac{u^5}{4}du$

$= \frac{u^6}{24}+c_1 = \frac{(1+2x^2)^6}{24}+c_1$

b $\frac{dy}{dx} = x(1+2x^2)^5\cos^2 2y$

$\Rightarrow \int\sec^2 2y\,dy = \int x(1+2x^2)^5dx$

$\Rightarrow \frac{1}{2}\tan 2y = \frac{(1+2x^2)^6}{24}+c_2$

$y=\frac{\pi}{8}, x=0 \Rightarrow \frac{1}{2} = \frac{1}{24}+c_2 \Rightarrow c_2 = \frac{11}{24}$

$\therefore \tan 2y = \frac{(1+2x^2)^6}{12}+\frac{11}{12}$

28 $I = \int\frac{1}{1+x^2}\,dx$

Let $x = \tan u \Rightarrow \frac{dx}{du} = \sec^2 u$

$I = \int\frac{1}{1+\tan^2 u}\sec^2 u\,du$

But $1+\tan^2 u = \sec^2 u$

So $I = \int du = u+c$

$= \arctan x + c$

29

$$x(x+2)\frac{dy}{dx}=y$$

$$\Rightarrow \int \frac{1}{y}\,dy=\int \frac{1}{x(x+2)}\,dx$$

$$\frac{1}{x(x+2)}\equiv \frac{A}{x}+\frac{B}{x+2}$$

$$\Rightarrow 1\equiv A(x+2)+Bx$$

$$x=0\Rightarrow 1=2A\Rightarrow A=\tfrac{1}{2}$$

$$x=-2\Rightarrow 1=-2B\Rightarrow B=-\tfrac{1}{2}$$

$$\text{So } \ln y=\int\left(\frac{\left(\frac{1}{2}\right)}{x}-\frac{\left(\frac{1}{2}\right)}{x+2}\right)dx$$

$$=\tfrac{1}{2}\ln|x|-\tfrac{1}{2}\ln|x+2|+c$$

$$\therefore y=\sqrt{\frac{kx}{x+2}}\quad \left(c=\tfrac{1}{2}\ln k\right)$$

$$x=2,y=2\Rightarrow 2=\sqrt{\frac{2k}{4}}\Rightarrow 4\times 2=k$$

$$\therefore y=\sqrt{\frac{8x}{x+2}}\quad \text{or}\quad y^2=\frac{8x}{x+2}$$

30 a $A=\pi r^2\Rightarrow \dfrac{dA}{dr}=2\pi r$

$$\frac{dr}{dt}=\frac{dr}{dA}\times\frac{dA}{dt}=\frac{1}{2\pi r}\times k\sin\left(\frac{t}{3\pi}\right)$$

$$\frac{dr}{dt}=\frac{k}{2\pi r}\sin\left(\frac{t}{3\pi}\right)$$

b $\displaystyle\int 2\pi r\ dr=\int k\sin\left(\frac{t}{3\pi}\right)dt$

$$\pi r^2=-3\pi k\cos\left(\frac{t}{3\pi}\right)+c$$

$$r^2=-3k\cos\left(\frac{t}{3\pi}\right)+c$$

$$r=1,t=0\Rightarrow 1=-3k+c\Rightarrow c=3k+1$$

$$r=2,t=\pi^2\Rightarrow 4=-\frac{3k}{2}+3k+1$$

$$\text{So } r^2=-6\cos\left(\frac{t}{3\pi}\right)+6+1$$

$$r^2=-6\cos\left(\frac{t}{3\pi}\right)+7$$

30 c $r=1.5\Rightarrow 2.25=-6\cos\left(\dfrac{t}{3\pi}\right)+7$

$$6\cos\left(\frac{t}{3\pi}\right)=4.75$$

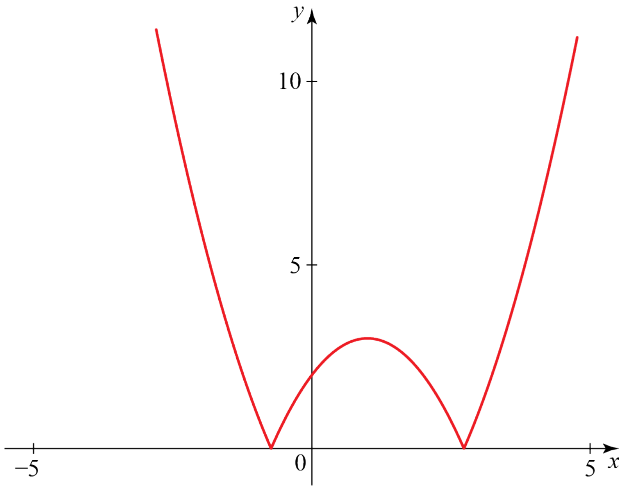
$$\cos\left(\frac{t}{3\pi}\right)\approx 0.7917$$

$$\frac{t}{3\pi}=0.6527$$

$$t=6.19\text{ days}$$

Challenge

a



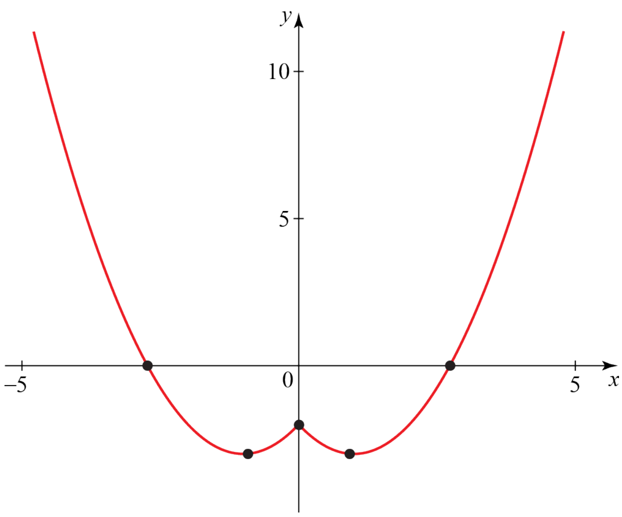
$$\int_{-3}^{3}|f(x)|\ dx = \int_{-3}^{3} f(x)\ dx + 2\times\left|\int_{-1}^{2} f(x)\ dx\right|$$

$$\int_{-3}^{3} f(x)\ dx = \left[\frac{x^3}{3}-\frac{x^2}{2}-2x\right]_{-3}^{3} = 6$$

$$\int_{-1}^{2} f(x)\ dx = \left[\frac{x^3}{3}-\frac{x^2}{2}-2x\right]_{-1}^{2} = \frac{9}{2}$$

$$\int_{-3}^{3}|f(x)|\ dx = 6+2\times\frac{9}{2} = 15$$

b



$$\int_{-3}^{3} f(|x|)\ dx = 2\times\int_{0}^{3} f(x)\ dx$$

$$\int_{0}^{3} f(x)\ dx = \left[\frac{x^3}{3}-\frac{x^2}{2}-2x\right]_{0}^{3} = -\frac{3}{2}$$

$$\int_{-3}^{3} f(|x|)\ dx = 2\times\left(-\frac{3}{2}\right) = -3$$