

Exercise 6G

$$1 \text{ a } \frac{dP}{dt} = kP$$

$$\int \frac{1}{P} dP = kt + c$$

$$\ln P = kt + c$$

$$t = 0, P = 200 \Rightarrow c = \ln 200$$

$$\ln P - \ln 200 = kt$$

$$\ln\left(\frac{P}{200}\right) = kt$$

$$\frac{P}{200} = e^{kt}$$

$$P = 200e^{kt}$$

$$b \quad k = 3: \quad 4000 = 200e^{3t}$$

$$e^{3t} = 20$$

$$t = \frac{1}{3} \ln 20 \approx 1 \text{ year}$$

- c The population could not increase in size in this way forever due to limitations such as available food or space.

$$2 \text{ a } \frac{dM}{dt} = M - M^2$$

$$\Rightarrow \int \frac{1}{M(1-M)} dM = \int 1 dt$$

$$\text{but } \frac{1}{M(1-M)} \equiv \frac{A}{M} + \frac{B}{1-M}$$

$$\therefore 1 \equiv A(1-M) + BM$$

$$M = 0: 1 = 1A, A = 1$$

$$M = 1: 1 = 1B, B = 1$$

$$\Rightarrow \int \left(\frac{1}{M} + \frac{1}{1-M} \right) dM = \int 1 dt$$

$$\Rightarrow \ln|M| - \ln|1-M| = t + c$$

$$\Rightarrow \ln \left| \frac{M}{1-M} \right| = t + c$$

$$\Rightarrow \frac{M}{1-M} = Ae^t$$

$$\text{When } t = 0, M = 0.5 \Rightarrow \frac{0.5}{0.5} = A^0 \Rightarrow A = 1$$

$$\therefore M = e^t - e^t M \Rightarrow M = \frac{e^t}{1+e^t}$$

$$b \quad t = \ln 2 \Rightarrow M = \frac{e^{\ln 2}}{1+e^{\ln 2}} = \frac{2}{1+2} = \frac{2}{3}$$

$$c \quad t \rightarrow \infty \Rightarrow M = \frac{1}{e^{-t} + 1} \rightarrow \frac{1}{1} = 1$$

So as t increases, M tends to 1.

$$3 \text{ a } \frac{dx}{dt} \propto \frac{1}{x^2} \Rightarrow \frac{dx}{dt} = \frac{k}{x^2}$$

$$\int x^2 dx = kt$$

$$\frac{x^3}{3} = kt + c$$

$$t = 0, x = 1 \Rightarrow c = \frac{1}{3}$$

$$x^3 = 3kt + 1$$

$$t = 20, x = 2 \Rightarrow 8 = 60k + 1 \Rightarrow k = \frac{7}{60}$$

$$x^3 = \frac{7}{20}t + 1$$

$$x = \sqrt[3]{\frac{7}{20}t + 1}$$

$$b \quad x = 3 \Rightarrow 27 = \frac{7}{20}t + 1$$

$$t = \frac{520}{7} = 74.3 \text{ days}$$

So time taken to go from 2 cm to 3 cm is $74.3 - 20 = 54.3$ days.

$$4 \text{ a } \frac{dT}{dt} \propto -(T - 25)$$

$$\frac{dT}{dt} = -k(T - 25)$$

The difference in temperature is $T - 25$.

The tea is cooling, so there should be a negative sign. k has to be positive or the tea would be warming.

$$4 \text{ b } \frac{dT}{dt} = -k(T - 25)$$

$$\int \frac{1}{T - 25} dT = -kt + c$$

$$\ln|T - 25| = -kt + c$$

$$t = 0, T = 85 \Rightarrow c = \ln 60$$

$$\frac{T - 25}{60} = e^{-kt}$$

$$t = 10, T = 55 \Rightarrow \frac{30}{60} = e^{-10k}$$

$$\ln \frac{1}{2} = -10k \Rightarrow k = 0.0693$$

$$t = 15 \Rightarrow \frac{T - 25}{60} = e^{-0.0693 \times 15}$$

$$T = 60 \times e^{-0.0693 \times 15} + 25 = 46.2 \text{ }^\circ\text{C to 1 d.p.}$$

$$5 \text{ a } \frac{dA}{dt} = \frac{A^{\frac{3}{2}}}{10t^2}$$

$$\int A^{-\frac{3}{2}} dA = \int \frac{1}{10t^2} dt$$

$$-\frac{2}{\sqrt{A}} = -\frac{1}{10t} + c$$

$$t = 1, A = 1 \Rightarrow -2 = -\frac{1}{10} + c$$

$$c = -\frac{19}{10}$$

$$-\frac{2}{\sqrt{A}} = -\frac{1}{10t} - \frac{19}{10}$$

$$\frac{2}{\sqrt{A}} = \frac{1}{10t} + \frac{19}{10} = \frac{1 + 19t}{10t}$$

$$\sqrt{A} = \frac{20t}{1 + 19t}$$

$$A = \left(\frac{20t}{1 + 19t} \right)^2$$

$$b \text{ As } t \rightarrow \infty, A \rightarrow \left(\frac{20}{19} \right)^2 = \frac{400}{361} \text{ from below.}$$

$$6 \text{ a } \text{Volume } V = 6000h \Rightarrow \frac{dV}{dh} = 6000$$

$$\frac{dV}{dt} = 12000 - 500h \text{ as the tub is filling at}$$

the rate of $12000 \text{ cm}^3/\text{min}$ and losing water at the rate of $500h \text{ cm}^3/\text{min}$.

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} = \frac{1}{6000} (12000 - 500h)$$

$$\frac{dh}{dt} = \frac{1}{60} (120 - 5h)$$

$$60 \frac{dh}{dt} = 120 - 5h$$

$$b \text{ } 60 \frac{dh}{dt} = 120 - 5h$$

$$\int \frac{60}{120 - 5h} dh = t + c$$

$$-12 \ln(120 - 5h) = t + c$$

$$t = 0, h = 6 \Rightarrow c = -12 \ln 90$$

$$-12 \ln(120 - 5h) = t - 12 \ln 90$$

$$h = 10:$$

$$t = 12 \ln 90 - 12 \ln 70$$

$$t = 12 \ln \frac{9}{7}$$

$$7 \text{ a } \frac{1}{P(10000 - P)} = \frac{A}{P} + \frac{B}{10000 - P}$$

$$1 = A(10000 - P) + BP$$

$$P = 0 \Rightarrow A = \frac{1}{10000}$$

$$P = 10000 \Rightarrow B = \frac{1}{10000}$$

$$\frac{1}{P(10000 - P)} = \frac{\frac{1}{10000}}{P} + \frac{\frac{1}{10000}}{10000 - P}$$

$$\begin{aligned}
 7 \text{ b } \frac{dP}{dt} &= \frac{1}{200}P(10\,000 - P) \\
 \frac{1}{10\,000} \int \left(\frac{1}{P} + \frac{1}{10\,000 - P} \right) dP &= \frac{1}{200}t + c \\
 \frac{1}{10\,000} (\ln|P| - \ln|10\,000 - P|) &= \frac{1}{200}t + c \\
 \ln P - \ln|10\,000 - P| &= 50t + d \\
 t = 0, P = 2500 \Rightarrow d &= \ln\left(\frac{2500}{7500}\right) = \ln\frac{1}{3} \\
 \ln P - \ln|10\,000 - P| - \ln\frac{1}{3} &= 50t \\
 \ln\left|\frac{3P}{10\,000 - P}\right| &= 50t \\
 \frac{3P}{10\,000 - P} &= e^{50t} \\
 3P &= (10\,000 - P)e^{50t} \\
 3Pe^{-50t} + P &= 10\,000 \\
 P(3e^{-50t} + 1) &= 10\,000 \\
 P &= \frac{10\,000}{1 + 3e^{-50t}} \\
 a = 10\,000, b = 1, c = 3
 \end{aligned}$$

c Maximum number of deer is when

$$\frac{dP}{dt} = 0 \Rightarrow P = 0 \text{ or } 10\,000$$

So maximum population is 10 000.

$$\begin{aligned}
 8 \text{ a } \frac{dV}{dt} &= 40 - \frac{1}{4}V \\
 -4 \frac{dV}{dt} &= V - 160
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \int -\frac{4}{V-160} dV &= t + c \\
 -4 \ln|V-160| &= t + c \\
 t = 0, V = 5000 \Rightarrow c &= -4 \ln 4840 \\
 4 \ln 4840 - 4 \ln|V-160| &= t \\
 \ln\left|\frac{4840}{V-160}\right| &= \frac{t}{4} \\
 \frac{4840}{V-160} &= e^{\frac{t}{4}} \\
 \frac{V-160}{4840} &= e^{-\frac{t}{4}} \\
 V &= 160 + 4840e^{-\frac{t}{4}} \\
 a = 160, b = 4840
 \end{aligned}$$

$$\text{c } t \rightarrow \infty \Rightarrow V \rightarrow 160, \text{ as } e^{-\frac{t}{4}} \rightarrow 0$$

$$\begin{aligned}
 9 \text{ a } \frac{dR}{dt} &= -kR \\
 \ln|R| &= -kt + c \\
 t = 0, R = R_0 \Rightarrow c &= \ln R_0 \\
 \ln R - \ln R_0 &= -kt \\
 \frac{R}{R_0} &= e^{-kt} \\
 R &= R_0 e^{-kt}
 \end{aligned}$$

$$\begin{aligned}
 \text{b } t = 5730, R = \frac{R_0}{2} \Rightarrow \frac{R_0}{2} &= R_0 e^{-5730k} \\
 e^{5730k} &= 2 \\
 5730k &= \ln 2 \\
 k &= \frac{1}{5730} \ln 2
 \end{aligned}$$

$$\begin{aligned}
 \text{c } R = \frac{R_0}{10} \Rightarrow \frac{R_0}{10} &= R_0 e^{-kt} \\
 e^{kt} &= 10 \\
 t = \frac{1}{k} \ln 10 &= \frac{5730 \ln 10}{\ln 2} = 19\,035
 \end{aligned}$$