Exercise 6F

1 a $\frac{dy}{dx} = (1+y)(1-2x)$

$\Rightarrow \int \frac{1}{1+y} dy = \int (1-2x) dx$

$\Rightarrow \ln|1+y| = x - x^2 + c$

$\Rightarrow 1+y = e^{(x-x^2+c)}$

$\Rightarrow 1+y = A\, e^{x-x^2}, \quad (A = e^c)$

$\Rightarrow y = A\, e^{x-x^2} - 1$

b $\frac{dy}{dx} = y \tan x$

$\Rightarrow \int \frac{1}{y} dy = \int \tan x\, dx$

$\Rightarrow \ln|y| = \ln|\sec x| + c$

$\Rightarrow \ln|y| = \ln|k \sec x|, \quad (c = \ln k)$

$\Rightarrow y = k \sec x$

c $\cos^2 x \frac{dy}{dx} = y^2 \sin^2 x$

$\Rightarrow \int \frac{1}{y^2} dy = \int \frac{\sin^2 x}{\cos^2 x} dx$

$\Rightarrow \int \frac{1}{y^2} dy = \int \tan^2 x\, dx = \int (\sec^2 x - 1) dx$

$\Rightarrow -\frac{1}{y} = \tan x - x + c$

$\Rightarrow y = \frac{-1}{\tan x - x + c}$

d $\frac{dy}{dx} = 2e^{x-y} = 2e^x e^{-y}$

$\Rightarrow \int \frac{1}{e^{-y}} dy = \int 2e^x dx$

i.e. $\Rightarrow \int e^y dy = \int 2e^x dx$

$\Rightarrow e^y = 2e^x + c$

$\Rightarrow y = \ln(2e^x + c)$

2 a $\frac{dy}{dx} = \sin x \cos^2 x$

$\Rightarrow \int dy = \int \sin x \cos^2 x\, dx$

$\Rightarrow y = -\frac{\cos^3 x}{3} + c$

$y = 0, x = \frac{\pi}{3} \Rightarrow 0 = -\frac{\left(\frac{1}{8}\right)}{3} + c \Rightarrow c = \frac{1}{24}$

$\therefore\ y = \frac{1}{24} - \frac{1}{3}\cos^3 x$

b $\frac{dy}{dx} = \sec^2 x \sec^2 y$

$\Rightarrow \int \frac{1}{\sec^2 y} dy = \int \sec^2 x\, dx$

$\Rightarrow \int \cos^2 y\, dy = \int \sec^2 x\, dx$

$\Rightarrow \int \left(\frac{1}{2} + \frac{1}{2}\cos 2y\right) dy = \int \sec^2 x\, dx$

$\Rightarrow \frac{1}{2}y + \frac{1}{4}\sin 2y = \tan x + c$

or $\sin 2y + 2y = 4\tan x + k$

$y = 0, x = \frac{\pi}{4} \Rightarrow 0 = 4 + k \Rightarrow k = -4$

$\therefore \sin 2y + 2y = 4\tan x - 4$

c $\frac{dy}{dx} = 2\cos^2 y \cos^2 x$

$\Rightarrow \int \frac{1}{\cos^2 y} dy = \int 2\cos^2 x\, dx$

$\Rightarrow \int \sec^2 y\, dy = \int (1 + \cos 2x) dx$

$\Rightarrow \tan y = x + \frac{1}{2}\sin 2x + c$

$x = 0, \quad y = \frac{\pi}{4} \Rightarrow 1 = 0 + c$

$\therefore\ \tan y = x + \frac{1}{2}\sin 2x + 1$

2 d $\sin y \cos x \dfrac{dy}{dx} = \dfrac{\cos y}{\cos x}$

$\tan y \dfrac{dy}{dx} = \sec^2 x$

$\int \tan y \, dy = \int \sec^2 x \, dx$

$-\ln|\cos y| = \tan x + c$

$x = 0, y = 0 \Rightarrow 0 = 0 + c \Rightarrow c = 0$

$-\ln|\cos y| = \tan x$

$\cos y = e^{-\tan x}$

$y = \arccos\left(e^{-\tan x}\right)$

3 a $x^2 \dfrac{dy}{dx} = y + xy$

$x^2 \dfrac{dy}{dx} = y(1+x)$

$\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{1+x}{x^2} = \dfrac{1}{x^2} + \dfrac{1}{x}$

$\int \dfrac{1}{y} \, dy = \int \left(\dfrac{1}{x^2} + \dfrac{1}{x}\right) dx$

$\ln y = -\dfrac{1}{x} + \ln x + \ln A$

$y = e^{-\frac{1}{x} + \ln x + \ln A} = e^{-\frac{1}{x}} \times e^{\ln x} \times e^{\ln A}$

$y = Axe^{-\frac{1}{x}}$

b $y = e^4, x = -1 \Rightarrow e^4 = -Ae$

$A = -e^3$

$y = -e^3 xe^{-\frac{1}{x}} = -xe^{\left(\frac{3x-1}{x}\right)}$

4 $(2y + 2yx)\dfrac{dy}{dx} = 1 - y^2$

$2y(1+x)\dfrac{dy}{dx} = 1 - y^2$

$\int \dfrac{2y}{1-y^2} \, dy = \int \dfrac{1}{1+x} \, dx$

$\ln k - \ln|1-y^2| = \ln|1+x|$

$\ln\left|\dfrac{k}{1-y^2}\right| = \ln|1+x|$

$\dfrac{k}{1-y^2} = 1 + x$

$x = 0, y = 0 \Rightarrow k = 1$

$\dfrac{1}{1-y^2} = 1 + x$

$1 - y^2 = \dfrac{1}{x+1}$

$y^2 = 1 - \dfrac{1}{x+1} = \dfrac{x}{x+1}$

$y = \sqrt{\dfrac{x}{x+1}}$

5 $e^{x+y}\dfrac{dy}{dx} = 2x + xe^y$

$e^x e^y \dfrac{dy}{dx} = x(2 + e^y)$

$\dfrac{e^y}{2+e^y}\dfrac{dy}{dx} = xe^{-x}$

$\int \dfrac{e^y}{2+e^y} \, dy = \int xe^{-x} \, dx$

$\ln|2+e^y| = \int xe^{-x} \, dx$

Let $u = x \Rightarrow \dfrac{du}{dx} = 1$

$\dfrac{dv}{dx} = e^{-x} \Rightarrow v = -e^{-x}$

So $\ln|2+e^y| = -xe^{-x} + \int e^{-x} \, dx$

$\ln|2+e^y| = -xe^{-x} - e^{-x} + c$

6 $(1-x^2)\frac{dy}{dx}=xy+y$

$(1-x^2)\frac{dy}{dx}=y(x+1)$

$\frac{1}{y}\frac{dy}{dx}=\frac{x+1}{(1-x^2)}=\frac{x+1}{(1-x)(1+x)}=\frac{1}{1-x}$

$\int\frac{1}{y}\,dy=\int\frac{1}{1-x}dx$

$\ln y=\ln k-\ln|1-x|=\ln\left|\frac{k}{1-x}\right|$

$y=\frac{k}{1-x}$

$y=6, x=0.5\Rightarrow 6=2k\Rightarrow k=3$

$y=\frac{3}{1-x}$

7 $(1+x^2)\frac{dy}{dx}=x-xy^2$

$(1+x^2)\frac{dy}{dx}=x(1-y^2)$

$\int\frac{1}{1-y^2}\,dy=\int\frac{x}{1+x^2}\,dx$

$\frac{1}{2}\int\left(\frac{1}{1+y}+\frac{1}{1-y}\right)\,dy=\int\frac{x}{1+x^2}\,dx$

$\tfrac{1}{2}\left(\ln|1+y|-\ln|1-y|\right)=\tfrac{1}{2}\ln|1+x^2|+c$

$x=0, y=2\Rightarrow c=\frac{1}{2}\ln 3$

$\left(\ln|1+y|-\ln|1-y|\right)=\ln|3(1+x^2)|$

$\frac{1+y}{y-1}=3+3x^2$

$1+y=3y+3x^2y-3-3x^2$

$1+3+3x^2=3y+3x^2y-y$

$3(x^2+1)+1=y(3(x^2+1)-1)$

$y=\frac{3(1+x^2)+1}{3(1+x^2)-1}$

8 $\frac{dy}{dx}=xe^{-y}$

$\int e^y\,dy=\int x\,dx$

$e^y=\frac{x^2}{2}+c$

$x=4, y=\ln 2\Rightarrow 2=8+c\Rightarrow c=-6$

$e^y=\frac{x^2-12}{2}$

$y=\ln\left|\frac{x^2-12}{2}\right|$

9 $\frac{dy}{dx}=\cos^2 y+\cos 2x\cos^2 y$

$\frac{dy}{dx}=\cos^2 y(1+\cos 2x)$

$\int\sec^2 y\,dy=\int\left(1+\cos 2x\right)\,dx$

$\tan y=x+\frac{1}{2}\sin 2x+c$

$x=\frac{\pi}{4}, y=\frac{\pi}{4}\Rightarrow 1=\frac{\pi}{4}+\frac{1}{2}\sin\frac{\pi}{2}+c$

$c=\frac{1}{2}-\frac{\pi}{4}=\frac{2-\pi}{4}$

$\tan y=x+\frac{1}{2}\sin 2x+\frac{2-\pi}{4}$

10 $\frac{dy}{dx}=xy\sin x$

$\int\frac{1}{y}\,dy=\int x\sin x\,dx$

$\ln|y|=-x\cos x+\int\cos x\,dx$

$\ln|y|=-x\cos x+\sin x+c$

$x=\frac{\pi}{2}, y=1\Rightarrow c=-1$

$\ln|y|=-x\cos x+\sin x-1$

11 a $I=\int\frac{3x+4}{x}\,dx$

$I=\int 3+\frac{4}{x}\,dx=3x+4\ln x+c$

11 b $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x\sqrt{y} + 4\sqrt{y}}{x} = \sqrt{y}\,\frac{3x+4}{x}$

$\int \frac{1}{\sqrt{y}}\,\mathrm{d}y = 3x + 4\ln x + c$ (from **a**)

$2\sqrt{y} = 3x + 4\ln x + c$

$x = 1, y = 16 \Rightarrow 8 = 3 + c \Rightarrow c = 5$

$\sqrt{y} = \frac{3}{2}x + 2\ln x + \frac{5}{2}$

$y = \left(\frac{3}{2}x + 2\ln x + \frac{5}{2}\right)^2$

12 a $\frac{8x-18}{(3x-8)(x-2)} = \frac{A}{3x-8} + \frac{B}{x-2}$

$8x - 18 = A(x-2) + B(3x-8)$

$x = 2: -2 = -2B \Rightarrow B = 1$

$x = \frac{8}{3}: \frac{64}{3} - 18 = \frac{2}{3}A \Rightarrow A = 5$

$\frac{8x-18}{(3x-8)(x-2)} = \frac{5}{3x-8} + \frac{1}{x-2}$

b $(x-2)(3x-8)\frac{\mathrm{d}y}{\mathrm{d}x} = (8x-18)y$

$\int \frac{1}{y}\,\mathrm{d}y = \int \frac{8x-18}{(3x-8)(x-2)}\,\mathrm{d}x$

$\ln|y| = \int\left(\frac{5}{3x-8} + \frac{1}{x-2}\right)\mathrm{d}x$

$\ln|y| = \frac{5}{3}\ln|3x-8| + \ln|x-2| + c$

c $x = 3, y = 8 \Rightarrow c = \ln 8$

$\ln|y| = \ln|3x-8|^{\frac{5}{3}} + \ln|8x-2|$

$\ln|y| = \ln\left|(3x-8)^{\frac{5}{3}}(8x-2)\right|$

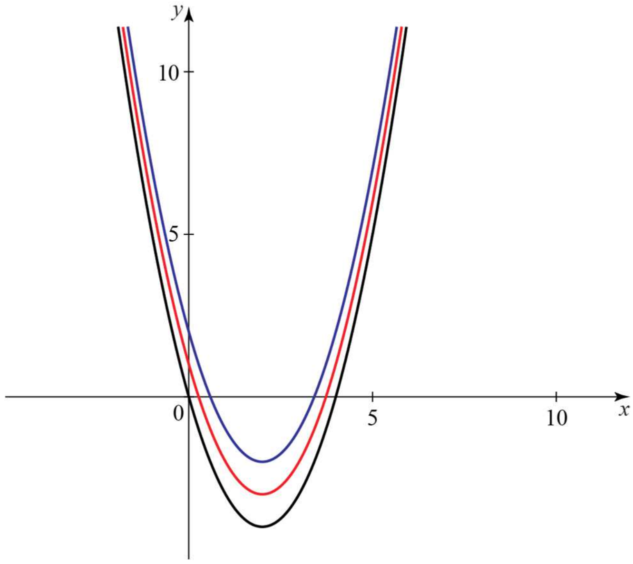
$y = (3x-8)^{\frac{5}{3}}(8x-2)$

$y = 8(x-2)(3x-8)^{\frac{5}{3}}$

13 a $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - 4$

$y = x^2 - 4x + c$

b



$y = x^2 - 4x, \quad y = x^2 - 4x + 1, \quad y = x^2 - 4x + 2$

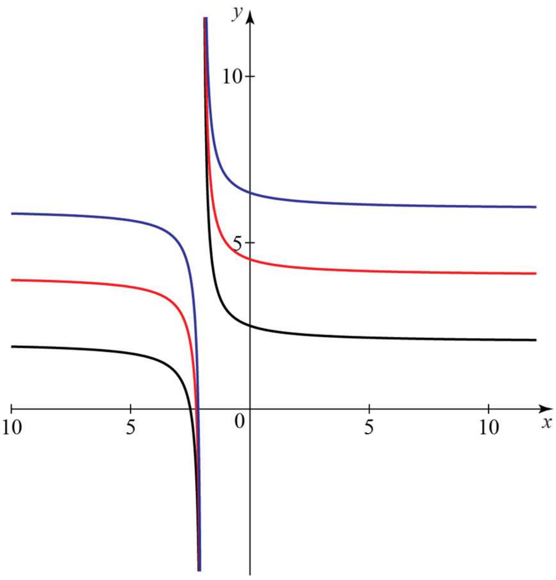
Any graphs of the form $y = x^2 - 4x + c$, where c is any real number.

14 a $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{(x+2)^2}$

$y = -\int \frac{1}{(x+2)^2}\,\mathrm{d}x$

$y = \frac{1}{x+2} + c$

b



$y = \frac{1}{x+2} + 2, \quad y = \frac{1}{x+2} + 4, \quad y = \frac{1}{x+2} + 6$

Any graphs of the form $y = \frac{1}{x+2} + c$, where c is any real number.

14 c $3.1 = \frac{1}{10} + c \Rightarrow c = 3$

$y = \frac{1}{x+2} + 3$

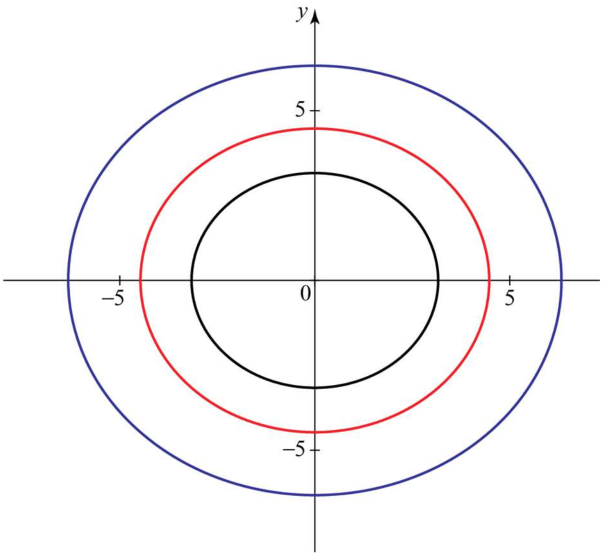
15 a $\frac{dy}{dx} = -\frac{x}{y}$

$\int y \, dy = \int x \, dx$

$\frac{y^2}{2} = -\frac{x^2}{2} + k$

$x^2 + y^2 = c$

b



$x^2 + y^2 = 10, \quad x^2 + y^2 = 20, \quad x^2 + y^2 = 40$

Circles with centre (0, 0) and radius $\sqrt{c}$
where c is any positive real number.

c $x^2 + y^2 = 49$