Exercise 6D

1 a $u = x \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = \sin x \Rightarrow v = -\cos x$

$\therefore \int x \sin x \, dx = -x\cos x - \int -\cos x \times 1 \, dx$

$= -x\cos x + \int \cos x \, dx$

$= -x\cos x + \sin x + c$

b $u = x \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = e^x \Rightarrow v = e^x$

$\therefore \int xe^x \, dx = xe^x - \int e^x \times 1 \, dx$

$= xe^x - e^x + c$

c $u = x \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = \sec^2 x \Rightarrow v = \tan x$

$\therefore \int x\sec^2 x \, dx = x\tan x - \int \tan x \times 1 \, dx$

$= x\tan x - \ln|\sec x| + c$

d $u = x \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = \sec x \tan x \Rightarrow v = \sec x$

$\therefore \int x\sec x \tan x \, dx = x\sec x - \int \sec x \times 1 \, dx$

$= x\sec x - \ln|\sec x + \tan x| + c$

e $u = x \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = \text{cosec}^2 x \Rightarrow v = -\cot x$

$\therefore \int \frac{x}{\sin^2 x} dx = \int x\,\text{cosec}^2 x \, dx$

$= -x\cot x - \int -\cot x \times 1 \, dx$

$= -x\cot x + \int \cot x \, dx$

$= -x\cot x + \ln|\sin x| + c$

2 a $u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = 3 \Rightarrow v = 3x$

$\therefore \int 3 \ln x \, dx = 3x \ln x - \int 3x \times \frac{1}{x} dx$

$= 3x \ln x - \int 3 \, dx$

$= 3x \ln x - 3x + c$

b $I = \int x \ln x \, dx$

Let $u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = x \Rightarrow v = \frac{x^2}{2}$

$I = \frac{x^2}{2}\ln x - \int \frac{x^2}{2} \times \frac{1}{x} \, dx$

$= \frac{x^2}{2}\ln x - \int \frac{x}{2} \, dx$

$= \frac{x^2}{2}\ln x - \frac{x^2}{4} + c$

c $u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = x^{-3} \Rightarrow v = \frac{x^{-2}}{-2}$

$\therefore \int \frac{\ln x}{x^3} dx = -\frac{1}{2x^2}\ln x - \int -\frac{1}{2x^2} \times \frac{1}{x} dx$

$= -\frac{\ln x}{2x^2} + \int \frac{1}{2} x^{-3} dx$

$= -\frac{\ln x}{2x^2} + \frac{x^{-2}}{2 \times (-2)} + c$

$= -\frac{\ln x}{2x^2} - \frac{1}{4x^2} + c$

2 d $u = (\ln x)^2 \Rightarrow \frac{du}{dx} = 2\ln x \times \frac{1}{x}$

$\frac{dv}{dx} = 1 \Rightarrow v = x$

$\therefore I = \int (\ln x)^2 dx = x(\ln x)^2$

$- \int x \times 2\ln x \times \frac{1}{x} dx$

$= x(\ln x)^2 - \int 2\ln x \, dx$

Let $J = \int 2\ln x \, dx$

$u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = 2 \Rightarrow v = 2x$

$\therefore J = 2x\ln x - \int 2x \times \frac{1}{x} dx = 2x\ln x - 2x + c$

$\therefore I = x(\ln x)^2 - 2x\ln x + 2x + c$

e $u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = x^2 + 1 \Rightarrow v = \frac{x^3}{3} + x$

$\therefore \int (x^2 + 1)\ln x \, dx = \ln x\left(\frac{x^3}{3} + x\right)$

$- \int \left(\frac{x^3}{3} + x\right) \times \frac{1}{x} dx$

$= \left(\frac{x^3}{3} + x\right)\ln x - \int \left(\frac{x^2}{3} + 1\right) dx$

$= \left(\frac{x^3}{3} + x\right)\ln x - \frac{x^3}{9} - x + c$

3 a $u = x^2 \Rightarrow \frac{du}{dx} = 2x$

$\frac{dv}{dx} = e^{-x} \Rightarrow v = -e^{-x}$

$\therefore I = \int x^2 e^{-x} dx = -x^2 e^{-x} - \int -e^{-x} \times 2x \, dx$

$= -x^2 e^{-x} + \int 2xe^{-x} \, dx$

Let $J = \int 2xe^{-x} \, dx$

$u = 2x \Rightarrow \frac{du}{dx} = 2$

$\frac{dv}{dx} = e^{-x} \Rightarrow v = -e^{-x}$

$\therefore J = -e^{-x} 2x - \int (-e^{-x}) \times 2dx$

$= 2xe^{-x} + \int 2e^{-x} \, dx$

$= -2xe^{-x} - 2e^{-x} + c$

$\therefore I = -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + c$

b $u = x^2 \Rightarrow \frac{du}{dx} = 2x$

$\frac{dv}{dx} = \cos x \Rightarrow v = \sin x$

$\therefore I = \int x^2 \cos x \, dx = x^2 \sin x - \int 2x \sin x \, dx$

Let $J = \int 2x \sin x \, dx$

$u = 2x \Rightarrow \frac{du}{dx} = 2$

$\frac{dv}{dx} = \sin x \Rightarrow v = -\cos x$

$\therefore J = -2x\cos x - \int (-\cos x) \times 2 \, dx$

$= -2x\cos x + \int 2\cos x \, dx$

$= -2x\cos x + 2\sin x + c$

$\therefore I = x^2 \sin x + 2x\cos x - 2\sin x + c$

3 c $u = 12x^2 \Rightarrow \frac{du}{dx} = 24x$

$\frac{dv}{dx} = (3+2x)^5 \Rightarrow v = \frac{(3+2x)^6}{12}$

$\therefore I = \int 12x^2(3+2x)^5\,dx = 12x^2\frac{(3+2x)^6}{12}$

$- \int 24x\frac{(3+2x)^6}{12}dx$

$= x^2(3+2x)^6 - \int 2x(3+2x)^6\,dx$

Let $J = \int 2x(3+2x)^6\,dx$

$u = 2x \Rightarrow \frac{du}{dx} = 2$

$v = \frac{(3+2x)^7}{14} \Rightarrow \frac{dv}{dx} = (3+2x)^6$

$\therefore J = 2x\frac{(3+2x)^7}{14} - \int\frac{(3+2x)^7}{14}\times 2\,dx$

$= x\frac{(3+2x)^7}{7} - \int\frac{(3+2x)^7}{7}dx$

$= x\frac{(3+2x)^7}{7} - \frac{(3+2x)^8}{7\times 16} + c$

$\therefore I = x^2(3+2x)^6 - x\frac{(3+2x)^7}{7} + \frac{(3+2x)^8}{112} + c$

d $u = 2x^2 \Rightarrow \frac{du}{dx} = 4x$

$\frac{dv}{dx} = \sin 2x \Rightarrow v = -\frac{1}{2}\cos 2x$

$\therefore I = \int 2x^2\sin 2x\,dx = -\frac{2x^2}{2}\cos 2x$

$-\int\left(-\frac{1}{2}\cos 2x\right)\times 4x\,dx$

$= -x^2\cos 2x + \int 2x\cos 2x\,dx$

Let $J = \int 2x\cos 2x\,dx$

$u = x \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = 2\cos 2x \Rightarrow v = \sin 2x$

$\therefore J = x\sin 2x - \int \sin 2x\,dx$

$= x\sin 2x + \frac{1}{2}\cos 2x + c$

$\therefore I = -x^2\cos 2x + x\sin 2x + \frac{1}{2}\cos 2x + c$

e $\int 2x^2\sec^2 x\tan x\,dx$

Let $u = 2x^2 \Rightarrow \frac{du}{dx} = 4x$

$\frac{dv}{dx} = \sec^2 x\tan x \Rightarrow v = \frac{1}{2}\sec^2 x$

$I = x^2\sec^2 x - 2\int x\sec^2 x\,dx$

Now let $u = x \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = \sec^2 x \Rightarrow v = \tan x$

$I = x^2\sec^2 x - 2(x\tan x - \int \tan x\,dx)$

$= x^2\sec^2 x - 2x\tan x + 2\ln|\sec x| + c$

4 a $u = x \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = e^{2x} \Rightarrow v = \frac{1}{2}e^{2x}$

$\therefore \int_0^{\ln 2} xe^{2x}\,dx = \left[\frac{1}{2}e^{2x}\times x\right]_0^{\ln 2} - \int_0^{\ln 2}\frac{1}{2}e^{2x}\,dx$

$= \left(\frac{1}{2}e^{2\ln 2}\ln 2\right) - (0) - \left[\frac{1}{4}e^{2x}\right]_0^{\ln 2}$

$= \frac{4}{2}\ln 2 - \left(\left(\frac{1}{4}e^{2\ln 2}\right) - \left(\frac{1}{4}e^0\right)\right)$

$= 2\ln 2 - \frac{4}{4} + \frac{1}{4}$

$= 2\ln 2 - \frac{3}{4}$

b $u = x \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = \sin x \Rightarrow v = -\cos x$

$\therefore \int_0^{\frac{\pi}{2}} x\sin x\,dx = [-x\cos x]_0^{\frac{\pi}{2}}$

$-\int_0^{\frac{\pi}{2}}(-\cos x)dx$

$= \left(-\frac{\pi}{2}\cos\frac{\pi}{2}\right) - (0) + \int_0^{\frac{\pi}{2}}(-\cos x)dx$

$= 0 + [\sin x]_0^{\frac{\pi}{2}}$

$= \left(\sin\frac{\pi}{2}\right) - (\sin 0)$

$= 1$

4 c $u = x \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = \cos x \Rightarrow v = \sin x$

$\therefore \int_0^{\frac{\pi}{2}} x\cos x\,dx = [x\sin x]_0^{\frac{\pi}{2}}$

$-\int_0^{\frac{\pi}{2}} \sin x\,dx$

$= \left(\frac{\pi}{2}\sin\frac{\pi}{2}\right) - (0) - [-\cos x]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{2} + \left(\cos\frac{\pi}{2}\right) - (\cos 0)$

$= \frac{\pi}{2} - 1$

d $u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = x^{-2} \Rightarrow v = -x^{-1}$

$\therefore \int_1^2 \frac{\ln x}{x^2}\,dx = \left[-\frac{\ln x}{x}\right]_1^2 - \int_1^2 \frac{1}{x}\times\left(-x^{-1}\right)dx$

$= \left(-\frac{\ln 2}{2}\right) - \left(-\frac{\ln 1}{1}\right) + \int_1^2 \frac{1}{x^2}\,dx$

$= -\frac{1}{2}\ln 2 + [-x^{-1}]_1^2$

$= -\frac{1}{2}\ln 2 + \left(-\frac{1}{2}\right) - \left(-\frac{1}{1}\right)$

$= \frac{1}{2}(1 - \ln 2)$

e $u = x \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = 4(1+x)^3 \Rightarrow v = (1+x)^4$

$\therefore \int_0^1 4x(1+x)^3\,dx = [x(1+x)^4]_0^1 - \int_0^1 (1+x)^4\,dx$

$= (1\times 2^4) - (0) - \left[\frac{(1+x)^5}{5}\right]_0^1$

$= 16 - \left(\left(\frac{2^5}{5}\right) - \left(\frac{1}{5}\right)\right)$

$= 16 - \frac{31}{5}$

$= 16 - 6.2$

$= 9.8$

f $\int_0^{\pi} x\cos\frac{1}{4}x\,dx$

Let $u = x \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = \cos\frac{1}{4}x \Rightarrow v = 4\sin\frac{1}{4}x$

$I = \left[4x\sin\frac{1}{4}x\right]_0^{\pi} - 4\int_0^{\pi}\sin\frac{1}{4}x\,dx$

$= \frac{4\pi}{\sqrt{2}} - 4\left[-4\cos\frac{1}{4}x\right]_0^{\pi}$

$= \frac{4\pi}{\sqrt{2}} - 4\left(-4\cos\frac{\pi}{4} + 4\right)$

$= 2\sqrt{2}\pi + 8\sqrt{2} - 16$

g $u = \ln|\sec x| \Rightarrow \frac{du}{dx} = \tan x$

$\frac{dv}{dx} = \sin x \Rightarrow v = -\cos x$

$\therefore \int_0^{\frac{\pi}{3}} \sin x \ln|\sec x|\,dx = \left[-\cos x\ \ln|\sec x|\right]_0^{\frac{\pi}{3}}$

$+\int_0^{\frac{\pi}{3}} \cos x\tan x\,dx$

$= \left(-\cos\frac{\pi}{3}\ln\left|\sec\frac{\pi}{3}\right|\right) - (-\cos 0\ \ln|\sec 0|)$

$+\int_0^{\frac{\pi}{3}} \sin x\,dx$

$= -\frac{1}{2}\ln 2 + 0 + [-\cos]_0^{\frac{\pi}{3}}$

$= -\frac{1}{2}\ln 2 + \left(-\frac{1}{2}\right) - (-1) = -\frac{1}{2}\ln 2 + \frac{1}{2}$

5 a $I = \int x\cos 4x\,dx$

Let $u = x \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = \cos 4x \Rightarrow v = \frac{1}{4}\sin x$

$I = \frac{x}{4}\sin 4x - \int \frac{1}{4}\sin 4x\,dx$

$I = \frac{x}{4}\sin 4x + \frac{1}{16}\cos 4x + c$

$I = \frac{1}{16}\left(4x\sin 4x + \cos 4x\right) + c$

5 b $I = \int x^2 \sin 4x \, dx$

Let $u = x^2 \Rightarrow \frac{du}{dx} = 2x$

$\frac{dv}{dx} = \sin 4x \Rightarrow v = -\frac{1}{4}\cos 4x$

$I = -\frac{x^2}{4}\cos 4x + \int \frac{x}{2}\cos 4x \, dx$

$= -\frac{x^2}{4}\cos 4x + \frac{1}{32}\left(4x\sin 4x + \cos 4x\right) + c$

$= \frac{1}{32}\left((1-8x^2)\cos 4x + 4x\sin 4x\right) + c$

6 a $\int \sqrt{8-x} \, dx = \int (8-x)^{\frac{1}{2}} dx$

$= -\tfrac{2}{3}(8-x)^{\frac{3}{2}} + c$

b $I = \int (x-2)\sqrt{8-x} \, dx$

Let $u = x-2 \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = \sqrt{8-x} \Rightarrow v = -\frac{2}{3}(8-x)^{\frac{3}{2}}$

$I = -\frac{2}{3}(x-2)(8-x)^{\frac{3}{2}} + \frac{2}{3}\int (8-x)^{\frac{3}{2}} \, dx$

$= -\frac{2}{3}(x-2)(8-x)^{\frac{3}{2}} - \frac{4}{15}(8-x)^{\frac{5}{2}} + c$

$= -\frac{2}{3}(x-2)(8-x)^{\frac{3}{2}} - \frac{4}{15}(8-x)^{\frac{3}{2}}(8-x) + c$

$= \frac{2}{3}(2-x)(8-x)^{\frac{3}{2}} + \frac{4}{15}(8-x)^{\frac{3}{2}}(x-8) + c$

$= \frac{2}{15}(8-x)^{\frac{3}{2}}(5(2-x) + 2(x-8)) + c$

$= \frac{2}{15}(8-x)^{\frac{3}{2}}(-3x-6) + c$

$= -\frac{2}{5}(8-x)^{\frac{3}{2}}(x+2) + c$

c $\int_4^7 (x-2)\sqrt{8-x} \, dx$

$= \left[-\frac{2}{5}(8-x)^{\frac{3}{2}}(x+2)\right]_4^7$

$= -\frac{2}{5} \times 9 + \frac{2}{5} \times 48$

$= \frac{78}{5} = 15.6$

7 a $\int \sec^2 3x \, dx$

$= \tfrac{1}{3}\tan 3x + c$

b $I = \int x\sec^2 3x \, dx$

Let $u = x \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = \sec^2 3x \Rightarrow v = \frac{1}{3}\tan 3x$

$I = \frac{x}{3}\tan 3x - \frac{1}{3}\int \tan 3x \, dx$

$I = \frac{x}{3}\tan 3x - \frac{1}{9}\ln\left|\sec 3x\right| + c$

c $\int_{\frac{\pi}{18}}^{\frac{\pi}{9}} x\sec^2 x \, dx$

$= \left[\frac{x}{3}\tan 3x - \frac{1}{9}\ln\left|\sec 3x\right|\right]_{\frac{\pi}{18}}^{\frac{\pi}{9}}$

$= \left(\frac{\pi}{27}\sqrt{3} - \frac{1}{9}\ln 2\right) - \left(\frac{\pi}{54} \times \frac{1}{\sqrt{3}} - \frac{1}{9}\ln\frac{2}{\sqrt{3}}\right)$

$= \left(\frac{\sqrt{3}\pi}{27} - \frac{1}{9}\ln 2\right) - \left(\frac{\sqrt{3}\pi}{162} - \frac{1}{9}\ln\frac{2}{\sqrt{3}}\right)$

$= \frac{5\sqrt{3}\pi}{162} - \frac{1}{9}\ln 2 + \frac{1}{9}\ln 2 - \frac{1}{9}\ln\sqrt{3}$

$= \frac{5\sqrt{3}\pi}{162} - \frac{1}{18}\ln 3$

$p = \frac{5\sqrt{3}}{162}, \quad q = \frac{1}{18}$