

Exercise 6C

1 a $\int x\sqrt{1+x} \, dx$

Let $u = 1+x$

$$\frac{du}{dx} = 1$$

So dx can be replaced by du .

$$\begin{aligned} \text{So } I &= \int (u-1)u^{\frac{1}{2}} \, du \\ &= \int (u^{\frac{3}{2}} - u^{\frac{1}{2}}) \, du \\ &= \frac{2u^{\frac{5}{2}}}{5} - \frac{2u^{\frac{3}{2}}}{3} + c \\ &= \frac{2(1+x)^{\frac{5}{2}}}{5} - \frac{2(1+x)^{\frac{3}{2}}}{3} + c \end{aligned}$$

b $\int \frac{1+\sin x}{\cos x} \, dx$

Let $u = \sin x$

$$\frac{du}{dx} = \cos x$$

So dx can be replaced by $\frac{du}{\cos x}$.

$$\begin{aligned} \text{So } I &= \int \frac{1+u}{\cos^2 x} \, du \\ &= \int \frac{1+u}{1-\sin^2 x} \, du \\ &= \int \frac{1+u}{1-u^2} \, du \\ &= \int \frac{1}{1-u} \, du \\ &= -\ln|1-u| + c \\ &= -\ln|1-\sin x| + c \end{aligned}$$

c $\int \sin^3 x \, dx$

Let $u = \cos x$

$$\frac{du}{dx} = -\sin x$$

So $\sin x dx$ can be replaced by $-du$.

Now $\sin^3 x = \sin x(1-\cos^2 x)$,

$$\begin{aligned} \text{so } I &= \int \sin x(1-\cos^2 x) \, dx \\ &= \int (u^2-1) \, du \\ &= \frac{u^3}{3} - u + c \\ &= \frac{\cos^3 x}{3} - \cos x + c \end{aligned}$$

d $\int \frac{2}{\sqrt{x}(x-4)} \, dx$

Let $u = \sqrt{x}$

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$$

So $\frac{dx}{\sqrt{x}}$ can be replaced by $2du$.

$$\begin{aligned} \text{So } I &= \int \frac{4}{(u^2-4)} \, du \\ &= \int \frac{4}{(u-2)(u+2)} \, du \\ &= \int \left(\frac{1}{u-2} - \frac{1}{u+2} \right) \, du \\ &= \ln|u-2| - \ln|u+2| + c \\ &= \ln \left| \frac{\sqrt{x}-2}{\sqrt{x}+2} \right| + c \end{aligned}$$

e $\int \sec^2 x \tan x \sqrt{1+\tan x} \, dx$

Let $u^2 = 1+\tan x$

$$2u \frac{du}{dx} = \sec^2 x$$

So $\sec^2 x dx$ can be replaced by $2udu$.

$$\begin{aligned} \text{So } I &= \int 2u(u^2-1)u \, du \\ &= \int (2u^4 - 2u^2) \, du \\ &= \frac{2u^5}{5} - \frac{2u^3}{3} + c \\ &= \frac{2(1+\tan x)^{\frac{5}{2}}}{5} - \frac{2(1+\tan x)^{\frac{3}{2}}}{3} + c \end{aligned}$$

f $\int \sec^4 x \, dx$

Let $u = \tan x$

$$\frac{du}{dx} = \sec^2 x$$

So $\sec^2 x dx$ can be replaced by du .

Now $\sec^2 x = 1+\tan^2 x$

$$\begin{aligned} \text{So } I &= \int (1+\tan^2 x) \sec^2 x \, dx \\ &= \int (1+u^2) \, du \\ &= u + \frac{u^3}{3} + c \\ &= \tan x + \frac{\tan^3 x}{3} + c \end{aligned}$$

2 a $\int_0^5 x\sqrt{x+4} \, dx$

Let $u = x + 4$

$$\frac{du}{dx} = 1$$

So dx can be replaced by du .

x	u
5	9
0	4

$$\begin{aligned} \text{So } I &= \int_4^9 (u-4)\sqrt{u} \, du \\ &= \int_4^9 (u^{\frac{3}{2}} - 4u^{\frac{1}{2}}) \, du \\ &= \left[\frac{2}{5}u^{\frac{5}{2}} - \frac{8}{3}u^{\frac{3}{2}} \right]_4^9 \\ &= \left(\frac{486}{5} - \frac{216}{3} \right) - \left(\frac{64}{5} - \frac{64}{3} \right) \\ &= \frac{506}{15} \end{aligned}$$

b $\int_0^2 x(2+x)^3 \, dx$

Let $u = 2 + x$

$$\frac{du}{dx} = 1$$

So dx can be replaced by du .

x	u
2	4
0	2

$$\begin{aligned} \text{So } I &= \int_2^4 (u-2)u^3 \, du \\ &= \left[\frac{u^5}{5} - \frac{u^4}{2} \right]_2^4 \\ &= \left(\frac{1024}{5} - \frac{256}{2} \right) - \left(\frac{32}{5} - \frac{16}{2} \right) \\ &= \frac{768}{10} + \frac{16}{10} \\ &= \frac{784}{10} \\ &= \frac{392}{5} \end{aligned}$$

c $\int_0^{\frac{\pi}{2}} \sin x \sqrt{3 \cos x + 1} \, dx$

Let $u = \cos x$

$$\frac{du}{dx} = -\sin x$$

So $\sin x dx$ can be replaced by $-du$.

x	u
$\frac{\pi}{2}$	0
0	1

$$\text{So } I = \int_1^0 -(3u+1)^{\frac{1}{2}} \, du$$

Consider $y = (3u+1)^{\frac{3}{2}}$

$$\frac{dy}{du} = \frac{9}{2}(3u+1)^{\frac{1}{2}}$$

$$\begin{aligned} I &= \left[-\frac{2}{9}(3u+1)^{\frac{3}{2}} \right]_1^0 \\ &= -\frac{2}{9} + \frac{16}{9} = \frac{14}{9} \end{aligned}$$

d $\int_0^{\frac{\pi}{3}} \sec x \tan x \sqrt{\sec x + 2} \, dx$

Let $u = \sec x$

$$\frac{du}{dx} = \sec x \tan x$$

So $\sec x \tan x dx$ can be replaced by du .

x	u
$\frac{\pi}{3}$	2
0	1

$$\begin{aligned} \text{So } I &= \int_1^2 (u+2)^{\frac{1}{2}} \, du \\ &= \left[\frac{2}{3}(u+2)^{\frac{3}{2}} \right]_1^2 \\ &= \frac{16}{3} - 2\sqrt{3} \end{aligned}$$

$$2 \text{ e } \int_1^4 \frac{1}{\sqrt{x}(4x-1)} dx$$

$$\text{Let } u = \sqrt{x}$$

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$$

So $\frac{dx}{\sqrt{x}}$ can be replaced by $2du$.

x	u
4	2
1	1

$$\begin{aligned} \text{So } I &= \int_1^2 \frac{2}{(4u^2-1)} du \\ &= \int_1^2 \left(\frac{1}{2u-1} - \frac{1}{2u+1} \right) du \\ &= \left[\frac{1}{2} \ln|2u-1| - \frac{1}{2} \ln|2u+1| \right]_1^2 \\ &= \frac{1}{2} \ln 3 - \frac{1}{2} \ln 5 - \frac{1}{2} \ln 1 + \frac{1}{2} \ln 3 \\ &= \ln 3 - \frac{1}{2} \ln 5 \\ &= \frac{1}{2} \ln 9 - \frac{1}{2} \ln 5 \\ &= \frac{1}{2} \ln \frac{9}{5} \end{aligned}$$

$$3 \text{ a } \int x(3+2x)^5 dx$$

$$\text{Let } u = 3+2x$$

$$\frac{du}{dx} = 2$$

So $2dx$ can be replaced by du .

$$\begin{aligned} I &= \int \frac{(u-3)}{4} u^5 du \\ &= \int \frac{u^6 - 3u^5}{4} du \\ &= \frac{u^7}{28} - \frac{3u^6}{24} + c \\ &= \frac{(3+2x)^7}{28} - \frac{(3+2x)^6}{8} + c \end{aligned}$$

$$b \int \frac{x}{\sqrt{1+x}} dx$$

$$\text{Let } u = 1+x$$

$$\frac{du}{dx} = 1$$

So dx can be replaced by du .

$$I = \int \frac{(u-1)}{u^{\frac{1}{2}}} du$$

$$I = \int \left(u^{\frac{1}{2}} - u^{-\frac{1}{2}} \right) du$$

$$= \frac{2}{3} u^{\frac{3}{2}} - 2u^{\frac{1}{2}} + c$$

$$= \frac{2}{3} (1+x)^{\frac{3}{2}} - 2\sqrt{1+x} + c$$

$$c \int \frac{\sqrt{x^2+4}}{x} dx$$

$$\text{Let } u = \sqrt{x^2+4}$$

$$\frac{du}{dx} = x(x^2+4)^{-\frac{1}{2}} = \frac{x}{u}$$

$$I = \int \frac{u}{x} \times \frac{u}{x} du$$

$$= \int \frac{u^2}{x^2} du = \int \frac{u^2}{u^2-4} du$$

$$= \int \left(1 + \frac{1}{u-2} - \frac{1}{u+2} \right) du$$

$$= u + (\ln|u-2| - \ln|u+2|) + c$$

$$= \sqrt{x^2+4} + \ln \left| \frac{\sqrt{x^2+4}-2}{\sqrt{x^2+4}+2} \right| + c$$

4 a $\int_2^7 x\sqrt{2+x} \, dx$

Let $u = 2 + x$

$$\frac{du}{dx} = 1$$

So dx can be replaced by du .

x	u
7	9
2	4

$$\begin{aligned} \text{So } I &= \int_4^9 (u-2)\sqrt{u} \, du \\ &= \left[\frac{2}{5}u^{\frac{5}{2}} - \frac{4}{3}u^{\frac{3}{2}} \right]_4^9 \\ &= \left(\frac{486}{5} - \frac{108}{3} \right) - \left(\frac{64}{5} - \frac{32}{3} \right) \\ &= \frac{886}{15} \end{aligned}$$

b $\int_2^5 \frac{1}{1+\sqrt{x-1}} \, dx$

Let $u = \sqrt{x-1}$

$$\frac{du}{dx} = \frac{1}{2\sqrt{x-1}} = \frac{1}{2u}$$

So dx can be replaced by $2udu$.

x	u
5	2
2	1

$$\begin{aligned} \text{So } I &= \int_1^2 \frac{2u}{1+u} \, du \\ I &= \int_1^2 2 \left(1 - \frac{1}{1+u} \right) \, du \\ &= \left[2u - 2 \ln|1+u| \right]_1^2 \\ &= (4 - 2 \ln 3) - (2 - 2 \ln 2) \\ &= 2 + 2 \ln 2 - 2 \ln 3 \\ &= 2 + 2 \ln \frac{2}{3} \end{aligned}$$

c $\int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{1+\cos \theta} \, d\theta$

Let $u = 1 + \cos \theta$

$$\frac{du}{d\theta} = -\sin \theta$$

So $d\theta$ can be replaced by $\frac{du}{\sin \theta}$.

θ	u
$\frac{\pi}{2}$	1
1	2

$$\begin{aligned} \text{So } I &= -\int_2^1 \frac{2 \sin \theta \cos \theta}{u} \frac{du}{\sin \theta} \\ &= \int_2^1 \frac{2(1-u)}{u} \, du \\ &= \left[2 \ln|u| - 2u \right]_2^1 \\ &= -2 - (2 \ln 2 - 4) \\ &= 2 - 2 \ln 2 \end{aligned}$$

5 $\int_6^{20} \frac{8x}{\sqrt{4x+1}} \, dx$

Let $u^2 = 4x+1$

$$2u \frac{du}{dx} = 4$$

So dx can be replaced by $\frac{u}{2} du$.

x	u
20	9
6	5

$$\begin{aligned} \text{So } I &= \int_5^9 \frac{2(u^2-1)}{u} \frac{u}{2} \, du \\ &= \left[\frac{u^3}{3} - u \right]_5^9 \\ &= \left(\frac{792}{3} - 9 \right) - \left(\frac{125}{3} - 5 \right) \\ &= \frac{592}{3} \end{aligned}$$

$$6 \int_{\ln 3}^{\ln 4} \frac{e^{4x}}{e^x - 2} dx$$

$$\text{Let } u^2 = e^x - 2$$

$$2u \frac{du}{dx} = e^x = u^2 + 2$$

So dx can be replaced by $\frac{2u}{u^2 + 2} du$.

x	u
$\ln 4$	$\sqrt{2}$
$\ln 3$	1

$$I = \int_1^{\sqrt{2}} \frac{2(u^3 + 2)^3}{u} du$$

$$I = \int_1^{\sqrt{2}} \left(2u^5 + 12u^3 + 24u + \frac{16}{u} \right) du$$

$$= \left[\frac{1}{3}u^6 + 3u^4 + 12u^2 + 16 \ln|u| \right]_1^{\sqrt{2}}$$

$$= \frac{70}{3} + 8 \ln 2$$

$$a = 70, b = 3, c = 8, d = 2$$

$$7 \int_0^{\frac{\pi}{3}} \sin^3 x \cos^2 x dx$$

$$\text{Let } u = \cos x$$

$$\frac{du}{dx} = -\sin x$$

So dx can be replaced by $-\frac{du}{\sin x}$.

x	u
$\frac{\pi}{3}$	$\frac{1}{2}$
0	1

$$I = \int_1^{\frac{1}{2}} -(1 - u^2)u^2 du$$

$$I = \int_1^{\frac{1}{2}} (u^2 - 1)u^2 du$$

$$= \int_1^{\frac{1}{2}} (u^4 - u^2) du$$

$$= \left[\frac{u^5}{5} - \frac{u^3}{3} \right]_1^{\frac{1}{2}}$$

$$= \left(\frac{1}{160} - \frac{1}{24} \right) - \left(\frac{1}{5} - \frac{1}{3} \right)$$

$$= \frac{47}{480}$$

$$8 \quad I = \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} x^2 \sqrt{1-x^2} \, dx$$

$$\text{Let } x = \sin \theta \Rightarrow \frac{dx}{d\theta} = \cos \theta$$

x	θ
$\frac{\sqrt{3}}{2}$	$\frac{\pi}{3}$
$\frac{1}{2}$	$\frac{\pi}{6}$

$$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^2 \theta \cos^2 \theta \, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{4} \sin^2 2\theta \, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{8} (1 - \cos 4\theta) \, d\theta$$

$$= \left[\frac{1}{8} \left(\theta - \frac{1}{4} \sin 4\theta \right) \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \frac{1}{8} \left(\left(\frac{\pi}{3} + \frac{\sqrt{3}}{8} \right) - \left(\frac{\pi}{6} - \frac{\sqrt{3}}{8} \right) \right)$$

$$= \frac{1}{8} \left(\frac{\pi}{6} + \frac{\sqrt{3}}{4} \right)$$

$$= \frac{2\pi + 3\sqrt{3}}{96}$$

Challenge

$$\int \frac{1}{x^2 \sqrt{9-x^2}} \, dx$$

$$\text{Let } x = 3 \sin u$$

$$\frac{dx}{du} = 3 \cos u$$

So dx can be replaced by $3 \cos u \, du$.

$$\int \frac{3 \cos u}{9 \sin^2 u \sqrt{9-9 \sin^2 u}} \, du$$

$$= \int \frac{3 \cos u}{9 \sin^2 u \cdot 3 \cos u} \, du$$

$$= \frac{1}{9} \int \cos \sec^2 u \, du$$

$$= -\frac{1}{9} \cot u + c$$

$$I = -\frac{\cos u}{9 \sin u} + c$$

$$\cos u = \sqrt{1 - \sin^2 u} = \sqrt{1 - \frac{x^2}{9}} = \frac{\sqrt{9-x^2}}{3}$$

$$I = -\frac{\frac{\sqrt{9-x^2}}{3}}{\frac{3}{9x}} + c$$

$$= -\frac{\sqrt{9-x^2}}{9x} + c$$