Exercise 6B

1 a $V = \pi\int_0^2 \left(10x^2\right)^2 \, dx$

$= \pi\int_0^2 100x^4 \, dx$

$= \pi\left[20x^5\right]_0^2$

$= \pi(640 - 0)$

$= 640\pi$

b $V = \pi\int_3^5 (5-x)^2 \, dx$

$= \pi\int_3^5 \left(25 - 10x + x^2\right) dx$

$= \pi\left[25x - 5x^2 + \frac{x^3}{3}\right]_3^5$

$= \pi\left(\left(125 - 125 + \frac{125}{3}\right) - (75 - 45 + 9)\right)$

$= \pi\left(\frac{125}{3} - 39\right)$

$= \frac{8}{3}\pi$

c $V = \pi\int_2^{10} \left(\sqrt{x}\right)^2 \, dx$

$= \pi\int_2^{10} x \, dx$

$= \pi\left[\frac{x^2}{2}\right]_2^{10}$

$= \pi\left(\frac{100}{2} - \frac{4}{2}\right)$

$= 48\pi$

d $V = \pi\int_1^2 \left(1 + \frac{1}{x^2}\right)^2 \, dx$

$V = \pi\int_1^2 \left(1 + \frac{2}{x^2} + \frac{1}{x^4}\right) dx$

$= \pi\left[x - \frac{2}{x} - \frac{1}{3x^3}\right]_1^2$

$= \pi\left(\left(2 - 1 - \frac{1}{24}\right) - \left(1 - 2 - \frac{1}{3}\right)\right)$

$= \pi\left(\frac{23}{24} + \frac{4}{3}\right)$

$= \frac{55}{24}\pi$

2 $5 + 4x - x^2 = 0$

$(5 - x)(1 + x) = 0$

$x > 0 \Rightarrow x = 5$

$V = \pi\int_0^5 \left(5 + 4x - x^2\right)^2 \, dx$

$= \pi\int_0^5 \left(25 + 40x + 6x^2 - 8x^3 + x^4\right) dx$

$= \pi\left[25x + 20x^2 + 2x^3 - 2x^4 + \frac{x^5}{5}\right]_0^5$

$= \pi(125 + 500 + 250 - 1250 + 625)$

$= 250\pi$

3 $V = \pi\int_1^8 \left(3 - \sqrt[3]{x}\right)^2 \, dx$

$V = \pi\int_1^8 \left(9 - 6x^{\frac{1}{3}} + x^{\frac{2}{3}}\right) dx$

$= \pi\left[9x - \frac{9}{2}x^{\frac{4}{3}} + \frac{3}{5}x^{\frac{5}{3}}\right]_1^8$

$= \pi\left(\left(72 - 72 + \frac{96}{5}\right) - \left(9 - \frac{9}{2} + \frac{3}{5}\right)\right)$

$= \pi\left(\frac{96}{5} - \frac{51}{10}\right)$

$= \frac{141}{10}\pi$

4 $\sqrt{x+2} = 0 \Rightarrow x = -2$

$V = \pi\int_{-2}^2 \left(\sqrt{x+2}\right)^2 \, dx$

$= \pi\int_{-2}^2 (x + 2) \, dx$

$= \pi\left[\frac{x^2}{2} + 2x\right]_{-2}^2$

$= \pi\left((2 + 4) - (2 - 4)\right)$

$= 8\pi$

5 a $9x^{\frac{3}{2}} - 3x^{\frac{5}{2}} = 0$

$3x^{\frac{3}{2}}(3 - x) = 0$

$x = 0$ or $x = 3$

Coordinates of A are (3, 0)

5 b $V=\pi\int_0^3\left(9x^{\frac{3}{2}}-3x^{\frac{5}{2}}\right)^2\,\mathrm{d}x$

$=\pi\int_0^3\left(81x^3-54x^4+9x^5\right)\,\mathrm{d}x$

$=\pi\left[\frac{81}{4}x^4-\frac{54}{5}x^5+\frac{3}{2}x^6\right]_0^3$

$=\pi\left(\frac{6561}{4}-\frac{13122}{5}+\frac{2187}{2}\right)$

$=\frac{2187}{20}\pi$

6 $\frac{\sqrt{3x^4-3}}{x^3}=0\Rightarrow x=\pm1$

From the graph $x>0$ so C cuts the x-axis at $x=1$

$V=\pi\int_1^6\left(\frac{\sqrt{3x^4-3}}{x^3}\right)^2\,\mathrm{d}x$

$=\pi\int_1^6\left(\frac{3x^4-3}{x^6}\right)\,\mathrm{d}x$

$=\pi\int_1^6\left(\frac{3}{x^2}-\frac{3}{x^6}\right)\,\mathrm{d}x$

$=\pi\left[-\frac{3}{x}+\frac{3}{5x^5}\right]_1^6$

$=\pi\left(\left(-\frac{3}{6}+\frac{3}{38\,880}\right)-\left(-3+\frac{3}{5}\right)\right)$

$=5.97$ (3 s.f.)

7 $5y^2-x^3=2x-3$

$y^2=\frac{1}{5}\left(x^3+2x-3\right)$

$V=\frac{\pi}{5}\int_1^4\left(x^3+2x-3\right)\,\mathrm{d}x$

$=\frac{\pi}{5}\left[\frac{x^4}{4}+x^2-3x\right]_1^4$

$=\frac{\pi}{5}\left((64+16-12)-\left(\frac{1}{4}+1-3\right)\right)$

$=\frac{\pi}{5}\left(68+\frac{7}{4}\right)$

$=\frac{279}{20}\pi$

8 $V=\pi\int_a^2\left(x\sqrt{4-x^2}\right)^2\,\mathrm{d}x$

$=\pi\int_a^2 x^2\left(4-x^2\right)\,\mathrm{d}x$

$=\pi\int_a^2\left(4x^2-x^4\right)\,\mathrm{d}x$

$=\pi\left[\frac{4x^3}{3}-\frac{x^5}{5}\right]_a^2$

$=\pi\left(\left(\frac{32}{3}-\frac{32}{5}\right)-\left(\frac{4a^3}{3}-\frac{a^5}{5}\right)\right)$

$=\pi\left(\frac{64}{15}-\frac{20a^3-3a^5}{15}\right)$

$=\frac{\pi}{15}\left(64-20a^3+3a^5\right)$

But $V=\frac{657}{160}\pi$ so $64-20a^3+3a^5=\frac{15\times657}{160}$

$3a^5-20a^3+\frac{77}{32}=0$

For $0<a<2$,

try $a=1\Rightarrow-\frac{467}{32}\neq0$,

try $a=\frac{1}{2}\Rightarrow0$

Hence, $\left(a-\frac{1}{2}\right)$ is a factor of $3a^5-20a^3+\frac{77}{32}$

Hence, a solution of $3a^5-20a^3+\frac{77}{32}=0$ is $a=\frac{1}{2}$

9 Equation of line is $y=r$

$V=\pi\int_0^h r^2\,\mathrm{d}x$

$=\pi\left[r^2x\right]_0^h$

$=\pi r^2h$

10 $x = t^{\frac{3}{2}},\ y = t^{\frac{1}{2}},\ 1 \le t \le 3$

$$\frac{dx}{dt} = \frac{3}{2}t^{\frac{1}{2}} \Rightarrow dx = \frac{3}{2}t^{\frac{1}{2}}\ dt$$

$$V = \pi\int_1^3 y^2\ dx$$

$$= \pi\int_1^3 (t)\frac{3}{2}t^{\frac{1}{2}}\ dt$$

$$= \frac{3}{2}\pi\int_1^3 t^{\frac{3}{2}}\ dt$$

$$= \frac{3}{2}\pi\left[\frac{2}{5}t^{\frac{5}{2}}\right]_1^3$$

$$= \frac{3}{5}\pi\left(3^{\frac{5}{2}} - 1^{\frac{5}{2}}\right)$$

$$= \frac{3}{5}\pi\left(9\sqrt{3} - 1\right) \text{ or } 27.5 \text{ (3 s.f.)}$$

11 $x = t^2 + 1,\ y = \frac{3}{t},\ 2 \le t \le 3$

$$\frac{dx}{dt} = 2t \Rightarrow dx = 2t\ dt$$

$$V = \pi\int_2^3 y^2\ dx$$

$$= \pi\int_2^3 \left(\frac{9}{t^2}\right)2t\ dt$$

$$= 18\pi\int_2^3 \frac{1}{t}dt$$

$$= 18\pi\left[\ln|t|\right]_2^3$$

$$= 18\pi\left(\ln 3 - \ln 2\right)$$

$$= 18\pi\ln\left(\frac{3}{2}\right)$$

Challenge

$x^2 - 7x + 10 = 0$

$(x-2)(x-5) = 0 \Rightarrow x = 2 \text{ or } x = 5$

So the x coordinates of the points where the curve touches the x-axis are 2 and 5.

Splitting the region into 3 sections,

R_1, R_2 and R_3

R_1 for $1 \le x \le 2$

R_2 for $2 < x \le 5$

R_3 for $5 < x \le 6$

Volume V_1 is generated by R_1, etc.

$$V_1 = \pi\int_1^2 \left(x^2 - 7x + 10\right)^2\ dx$$

$$= \pi\int_1^2 \left(x^4 - 14x^3 + 69x^2 - 140x + 100\right)\ dx$$

$$= \pi\left[\frac{x^5}{5} - \frac{7x^4}{2} + 23x^3 - 70x^2 + 100x\right]_1^2$$

$$= \pi\left(\frac{272}{5} - \frac{497}{10}\right) = \frac{47}{10}\pi$$

$$V_2 = \pi\int_2^5 \left(x^2 - 7x + 10\right)^2\ dx$$

$$= \pi\int_2^5 \left(x^4 - 14x^3 + 69x^2 - 140x + 100\right)\ dx$$

$$= \pi\left[\frac{x^5}{5} - \frac{7x^4}{2} + 23x^3 - 70x^2 + 100x\right]_2^5$$

$$= \pi\left(\frac{125}{2} - \frac{272}{5}\right) = \frac{81}{10}\pi$$

$V_3 = V_1$ using the symmetry of the curve.

So total volume generated $= \pi\left(\frac{47}{10} + \frac{81}{10} + \frac{47}{10}\right)$

$$= \frac{175}{10}\pi$$

$$= \frac{35}{2}\pi$$