Exercise 6A

1 $x = t^2,\ y = t + 1,\ -1 \le t \le 2$

$$\frac{dx}{dt} = 2t \Rightarrow dx = 2t\ dt$$

$$A = \int_{-1}^{2} y\,dx$$

$$= \int_{-1}^{2} (t+1)2t\,dt$$

$$= 2\int_{-1}^{2} \left(t^2 + t\right) dt$$

$$= 2\left[\frac{1}{3}t^3 + \frac{1}{2}t^2\right]_{-1}^{2}$$

$$= 2\left[\left(\frac{1}{3}(2)^3 + \frac{1}{2}(2)^2\right) - \left(\frac{1}{3}(-1)^3 + \frac{1}{2}(-1)^2\right)\right]$$

$$= 2\left(\frac{8}{3} + 2 + \frac{1}{3} - \frac{1}{2}\right)$$

$$= 9$$

2 $x = \sqrt{t},\ y = 4t^{\frac{3}{2}},\ 3 \le t \le 5$

$$\frac{dx}{dt} = \frac{1}{2}t^{-\frac{1}{2}} \Rightarrow dx = \frac{1}{2}t^{-\frac{1}{2}}\,dt$$

$$A = \int_{3}^{5} y\,dx$$

$$= \int_{3}^{5} \left(4t^{\frac{3}{2}}\right)\frac{1}{2}t^{-\frac{1}{2}}\,dt$$

$$= 2\int_{3}^{5} t\,dt$$

$$= 2\left[\frac{1}{2}t^2\right]_{3}^{5}$$

$$= \left(5^2 - 3^2\right)$$

$$= 16$$

3 $x = \sin t,\ y = 2\cos t,\ 0 \le t \le \frac{\pi}{3}$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \cos t \Rightarrow \mathrm{d}x = \cos t\,\mathrm{d}t$$

$$A = \int_0^{\frac{\pi}{3}} y\,\mathrm{d}x$$

$$= \int_0^{\frac{\pi}{3}} (2\cos t)\cos t\,\mathrm{d}t$$

$$= 2\int_0^{\frac{\pi}{3}} (\cos^2 t)\,\mathrm{d}t$$

$$= \int_0^{\frac{\pi}{3}} (1 + \cos 2t)\,\mathrm{d}t$$

$$= \left[t + \frac{1}{2}\sin 2t\right]_0^{\frac{\pi}{3}}$$

$$= \left(\left(\frac{\pi}{3}\right) + \frac{1}{2}\sin 2\left(\frac{\pi}{3}\right)\right) - \left(0 + \frac{1}{2}\sin 2(0)\right)$$

$$= \frac{\pi}{3} + \frac{\sqrt{3}}{4}$$

4 $x = 6t^2,\ y = \frac{t}{4},\ 2 \le t \le b,$ area $= 117$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 12t \Rightarrow \mathrm{d}x = 12t\,\mathrm{d}t$$

$$A = \int_2^b y\,\mathrm{d}x$$

$$\int_2^b \left(\frac{t}{4}\right) 12t\,\mathrm{d}t = 117$$

$$\int_2^b 3t^2\,\mathrm{d}t = 117$$

$$\left[t^3\right]_2^b = 117$$

$$b^3 - 8 = 117$$

$$b^3 = 125$$

$$b = 5$$

5 $x = 2t^{\frac{1}{2}},\ y = 4t^{-\frac{1}{2}},\ 4 \le t \le 16$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = t^{-\frac{1}{2}} \Rightarrow \mathrm{d}x = t^{-\frac{1}{2}}\,\mathrm{d}t$$

$$A = \int_4^{16} y\,\mathrm{d}x$$

$$= \int_4^{16} \left(4t^{-\frac{1}{2}}\right) t^{-\frac{1}{2}}\,\mathrm{d}t$$

$$= \int_4^{16} \frac{4}{t}\,\mathrm{d}t$$

$$= \left[4\ln|t|\right]_4^{16}$$

$$= 4\left(\ln 16 - \ln 4\right)$$

$$= 4\ln 4$$

$$= \ln 256$$

6 $x = \mathrm{e}^{2t},\ y = \mathrm{e}^{3t},\ \ln 3 \le t \le \ln 8$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2\mathrm{e}^{2t} \Rightarrow \mathrm{d}x = 2\mathrm{e}^{2t}\,\mathrm{d}t$$

$$A = \int_{\ln 3}^{\ln 8} y\,\mathrm{d}x$$

$$= \int_{\ln 3}^{\ln 8} \left(\mathrm{e}^{3t}\right) 2\mathrm{e}^{2t}\,\mathrm{d}t$$

$$= 2\int_{\ln 3}^{\ln 8} \mathrm{e}^{5t}\,\mathrm{d}t$$

$$= 2\left[\frac{1}{5}\mathrm{e}^{5t}\right]_{\ln 3}^{\ln 8}$$

$$= \frac{2}{5}\left(\mathrm{e}^{5\ln 8} - \mathrm{e}^{5\ln 3}\right)$$

$$= \frac{2}{5}\left(\mathrm{e}^{\ln 8^5} - \mathrm{e}^{\ln 3^5}\right)$$

$$= \frac{2}{5}\left(8^5 - 3^5\right)$$

$$= 13\,010$$