Exercise 5B

1 $u = y^n$

$\frac{du}{dy} = ny^{n-1}$

$\frac{d(y^n)}{dx} = \frac{du}{dx} = \frac{du}{dy} \times \frac{dy}{dx} = ny^{n-1}\frac{dy}{dx}$

2 $\frac{d(xy)}{dx} = x\frac{d(y)}{dx} + \frac{d(x)}{dx}y = x\frac{d(y)}{dx} + 1 \times y$

$= x\frac{dy}{dx} + y$

3 a $x^2 + y^3 = 2$

Differentiate with respect to x:

$2x + 3y^2\frac{dy}{dx} = 0$

$3y^2\frac{dy}{dx} = -2x$

$\therefore \frac{dy}{dx} = -\frac{2x}{3y^2}$

b $x^2 + 5y^2 = 14$

$2x + 10y\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = \frac{-2x}{10y} = -\frac{x}{5y}$

c $x^2 + 6x - 8y + 5y^2 = 13$

$2x + 6 - 8\frac{dy}{dx} + 10y\frac{dy}{dx} = 0$

$2x + 6 = (8 - 10y)\frac{dy}{dx}$

$\therefore \frac{dy}{dx} = \frac{2x+6}{8-10y} = \frac{x+3}{4-5y}$

d $y^3 + 3x^2y - 4x = 0$

$3y^2\frac{dy}{dx} + \left(3x^2\frac{dy}{dx} + y \times 6x\right) - 4 = 0$

$(3y^2 + 3x^2)\frac{dy}{dx} = 4 - 6xy$

$\therefore \frac{dy}{dx} = \frac{4-6xy}{3(x^2+y^2)}$

e $3y^2 - 2y + 2xy = x^3$

$6y\frac{dy}{dx} - 2\frac{dy}{dx} + \left(2x\frac{dy}{dx} + y \times 2\right) = 3x^2$

$(6y - 2 + 2x)\frac{dy}{dx} = 3x^2 - 2y$

$\therefore \frac{dy}{dx} = \frac{3x^2 - 2y}{2x + 6y - 2}$

f $x = \frac{2y}{x^2 - y}$

$x^3 - xy = 2y$

$x^3 - xy - 2y = 0$

Differentiate with respect to x:

$3x^2 - \left(x\frac{dy}{dx} + y\right) - 2\frac{dy}{dx} = 0$

$3x^2 - y = (x+2)\frac{dy}{dx}$

$\therefore \frac{dy}{dx} = \frac{3x^2 - y}{x + 2}$

g $(x-y)^4 = x + y + 5$

Differentiate with respect to x

(using the chain rule on the first term):

$4(x-y)^3\left(1 - \frac{dy}{dx}\right) = 1 + \frac{dy}{dx}$

$4(x-y)^3 - 1 = \left(1 + 4(x-y)^3\right)\frac{dy}{dx}$

$\therefore \frac{dy}{dx} = \frac{4(x-y)^3 - 1}{1 + 4(x-y)^3}$

h $e^x y = xe^y$

$e^x\frac{dy}{dx} + ye^x = xe^y\frac{dy}{dx} + e^y \times 1$

$e^x\frac{dy}{dx} - xe^y\frac{dy}{dx} = e^y - ye^x$

$(e^x - xe^y)\frac{dy}{dx} = e^y - ye^x$

$\therefore \frac{dy}{dx} = \frac{e^y - ye^x}{e^x - xe^y}$

3 i $\sqrt{xy} + x + y^2 = 0$

$(xy)^{\frac{1}{2}} + x + y^2 = 0$

$\frac{1}{2}(xy)^{-\frac{1}{2}}\left(x\frac{dy}{dx} + y\right) + 1 + 2y\frac{dy}{dx} = 0$

Multiply both sides by $2\sqrt{xy}$:

$\left(x\frac{dy}{dx} + y\right) + 2\sqrt{xy} + 4y\sqrt{xy}\frac{dy}{dx} = 0$

$\left(x + 4y\sqrt{xy}\right)\frac{dy}{dx} = -2\sqrt{xy} - y$

$\therefore \frac{dy}{dx} = \frac{-2\sqrt{xy} - y}{x + 4y\sqrt{xy}}$

4 $x^2 + 3xy^2 - y^3 = 9$

Differentiate with respect to x:

$2x + \left(3x \times 2y\frac{dy}{dx} + y^2 \times 3\right) - 3y^2\frac{dy}{dx} = 0$

Substitute $x = 2$ and $y = 1$ to give

$4 + \left(12\frac{dy}{dx} + 3\right) - 3\frac{dy}{dx} = 0$

$9\frac{dy}{dx} = -7$

$\frac{dy}{dx} = -\frac{7}{9}$

$\therefore$ gradient of the tangent at (2, 1) is $-\frac{7}{9}$

Equation of the tangent is

$(y - 1) = -\frac{7}{9}(x - 2)$

$y = -\frac{7}{9}x + \frac{23}{9}$

or $7x + 9y - 23 = 0$

5 $(x + y)^3 = x^2 + y$

Differentiate with respect to x:

$3(x + y)^2\left(1 + \frac{dy}{dx}\right) = 2x + \frac{dy}{dx}$

Substitute $x = 1$ and $y = 0$ to give

$3\left(1 + \frac{dy}{dx}\right) = 2 + \frac{dy}{dx}$

$2\frac{dy}{dx} = -1 \quad \therefore \frac{dy}{dx} = \frac{-1}{2}$

$\therefore$ gradient of the normal at (1, 0) is 2.

Equation of the normal is

$y - 0 = 2(x - 1)$

or $y = 2x - 2$

6 $x^2 + 4y^2 - 6x - 16y + 21 = 0$

Differentiate with respect to x:

$2x + 8y\frac{dy}{dx} - 6 - 16\frac{dy}{dx} = 0$

$8y\frac{dy}{dx} - 16\frac{dy}{dx} = 6 - 2x$

$(8y - 16)\frac{dy}{dx} = 6 - 2x$

$\frac{dy}{dx} = \frac{6 - 2x}{8y - 16}$

For zero gradient:

$\frac{6 - 2x}{8y - 16} = 0$

$6 - 2x = 0$

$x = 3$

Substitute $x = 3$ into the equation of the curve to give

$9 + 4y^2 - 18 - 16y + 21 = 0$

$4y^2 - 16y + 12 = 0$

$y^2 - 4y + 3 = 0$

$(y - 1)(y - 3) = 0$

$y = 1$ or 3

$\therefore$ the coordinates of the points of zero gradient are (3, 1) and (3, 3).

7 $2x^2 + 3y^2 - x + 6xy + 5 = 0$

$4x + 6y\frac{dy}{dx} - 1 + 6\left(x\frac{dy}{dx} + y\right) = 0$

$(6y + 6x)\frac{dy}{dx} = 1 - 6y - 4x$

$\frac{dy}{dx} = \frac{1 - 6y - 4x}{6(x + y)}$

When $x = 1$ and $y = -2$,

$\frac{dy}{dx} = \frac{1 - 6(-2) - 4}{6(1 - 2)} = -\frac{3}{2}$

Equation of tangent at (1, −2) is

$y - (-2) = -\frac{3}{2}(x - 1)$

$2y + 4 = -3x + 3$

$3x + 2y + 1 = 0$

8 $3^x = y - 2xy$

$3^x \ln 3 = \frac{dy}{dx} - 2\left(x\frac{dy}{dx} + y\right)$

$3^x \ln 3 + 2y = (1-2x)\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{3^x \ln 3 + 2y}{1-2x}$

Substitute $x = 2$ and $y = -3$ to give

$\frac{dy}{dx} = \frac{3^2 \ln 3 - 6}{1-4} = 2 - 3\ln 3$

9 $\ln(y^2) = \frac{1}{2}x\ln(x-1)$

$2\ln y = \frac{1}{2}x\ln(x-1)$

$\frac{2}{y}\frac{dy}{dx} = \frac{1}{2}\left(\left(x \times \frac{1}{x-1}\right) + \ln(x-1)\right)$

$\frac{dy}{dx} = \frac{y}{4}\left(\frac{x}{x-1} + \ln(x-1)\right)$

When $x = 4$,
the equation of the curve gives
$\ln(y^2) = 2\ln 3 = \ln 9 \Rightarrow y^2 = 9$

$\therefore y = 3$ (because $y > 0$)

Hence $\frac{dy}{dx} = \frac{3}{4}\left(\frac{4}{3} + \ln 3\right) = 1 + \frac{3}{4}\ln 3$

10 a $\sin x + \cos y = 0.5$

$\cos x - \sin y\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = \frac{\cos x}{\sin y}$

b At stationary points $\frac{dy}{dx} = 0$

$\frac{\cos x}{\sin y} = 0$ when $\cos x = 0$

$\therefore x = \pm\frac{\pi}{2}$ (in the interval $-\pi < x < \pi$)

When $x = \frac{\pi}{2}$, $1 + \cos y = 0.5$

$\cos y = -0.5 \Rightarrow y = \pm\frac{2\pi}{3}$

When $x = -\frac{\pi}{2}$, $-1 + \cos y = 0.5$

$\cos y = 1.5 \Rightarrow$ no solutions

Therefore the stationary points are
$\left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$ and $\left(\frac{\pi}{2}, -\frac{2\pi}{3}\right)$.

11 a $ye^{-3x} - 3x = y^2$

$y(-3e^{-3x}) + e^{-3x}\frac{dy}{dx} - 3 = 2y\frac{dy}{dx}$

$(e^{-3x} - 2y)\frac{dy}{dx} = 3(ye^{-3x} + 1)$

$\frac{dy}{dx} = \frac{3(ye^{-3x} + 1)}{e^{-3x} - 2y}$

b Substitute $x = 0$ and $y = 0$ to give

$\frac{dy}{dx} = \frac{3(0 \times e^0 + 1)}{e^0 - 2 \times 0} = 3$

Equation of tangent at (0, 0) is

$y - 0 = 3(x - 0)$

$y = 3x$

Challenge

a $6x+y^2+2xy=x^2$

$6+2y\frac{dy}{dx}+2\left(x\frac{dy}{dx}+y\right)=2x$

$(2y+2x)\frac{dy}{dx}=2x-2y-6$

$\therefore \frac{dy}{dx}=\frac{2x-2y-6}{2y+2x}=\frac{x-y-3}{y+x}$

$\frac{dy}{dx}=0$ only when $x-y-3=0$

or $y=x-3$

Substitute $y=x-3$ into the equation of the curve to give

$6x+(x-3)^2+2x(x-3)=x^2$

$2x^2-6x+9=0$

The discriminant of this quadratic is

$(-6)^2-4\times2\times9=-36<0$

so there are no real solutions.

Hence there are no points on C such that

$\frac{dy}{dx}=0.$

b $\frac{dx}{dy}=\frac{y+x}{x-y-3}$

$\frac{dx}{dy}=0$ when $y+x=0$

or $y=-x$

Substitute $y=-x$ into the equation of the curve to give

$6x+(-x)^2+2x(-x)=x^2$

$2x^2-6x=0$

$x(x-3)=0$

$x=0$ or $x=3$

So $y=0$ or $y=-3$

Therefore the coordinates of the points on C such that $\frac{dx}{dy}=0$ are (0, 0) and (3, −3).