Exercise 5A

1 a $x = 2t,\ y = t^2 - 3t + 2$

$\frac{dx}{dt} = 2,\ \frac{dy}{dt} = 2t - 3$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t-3}{2} = t - \frac{3}{2}$

b $x = 3t^2,\ y = 2t^3$

$\frac{dx}{dt} = 6t,\ \frac{dy}{dt} = 6t^2$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6t^2}{6t} = t$

c $x = t + 3t^2,\ y = 4t$

$\frac{dx}{dt} = 1 + 6t,\ \frac{dy}{dt} = 4$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4}{1+6t}$

d $x = t^2 - 2,\ y = 3t^5$

$\frac{dx}{dt} = 2t,\ \frac{dy}{dt} = 15t^4$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{15t^4}{2t} = \frac{15t^3}{2}$

e $x = \frac{2}{t},\ y = 3t^2 - 2$

$\frac{dx}{dt} = -\frac{2}{t^2},\ \frac{dy}{dt} = 6t$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{6t}{\frac{2}{t^2}} = -3t^3$

f $x = \frac{1}{2t-1},\ y = \frac{t^2}{2t-1}$

$\frac{dx}{dt} = -\frac{2}{(2t-1)^2},\ \frac{dy}{dt} = \frac{2t(2t-1) - 2t^2}{(2t-1)^2}$

$= \frac{2t^2 - 2t}{(2t-1)^2}$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{2t^2 - 2t}{2} = t(1-t)$

g $x = \frac{2t}{1+t^2},\ y = \frac{1-t^2}{1+t^2}$

$\frac{dx}{dt} = \frac{2(1+t^2) - 4t^2}{(1+t^2)^2} = \frac{2(1-t^2)}{(1+t^2)^2}$

$\frac{dy}{dt} = \frac{-2t(1+t^2) - 2t(1-t^2)}{(1+t^2)^2} = -\frac{4t}{(1+t^2)^2}$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-4t}{2(1-t^2)} = \frac{2t}{t^2-1}$

h $x = t^2 e^t,\ y = 2t$

$\frac{dx}{dt} = t^2 e^t + 2te^t,\ \frac{dy}{dt} = 2$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2}{t^2 e^t + 2te^t} = \frac{2}{(t^2 + 2t)e^t}$

i $x = 4\sin 3t,\ y = 3\cos 3t$

$\frac{dx}{dt} = 12\cos 3t,\ \frac{dy}{dt} = -9\sin 3t$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-9\sin 3t}{12\cos 3t} = -\frac{3}{4}\tan 3t$

j $x = 2 + \sin t,\ y = 3 - 4\cos t$

$\frac{dx}{dt} = \cos t,\ \frac{dy}{dt} = 4\sin t$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4\sin t}{\cos t} = 4\tan t$

1 k $x = \sec t,\ y = \tan t$

$\frac{dx}{dt} = \sec t \tan t,\ \frac{dy}{dt} = \sec^2 t$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\sec^2 t}{\sec t \tan t} = \frac{\sec t}{\tan t} = \operatorname{cosec} t$

l $x = 2t - \sin 2t,\ y = 1 - \cos 2t$

$\frac{dx}{dt} = 2 - 2\cos 2t,\ \frac{dy}{dt} = 2\sin 2t$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2\sin 2t}{2 - 2\cos 2t} = \frac{\sin 2t}{1 - \cos 2t}$

$= \frac{2\sin t \cos t}{1 - (1 - 2\sin^2 t)} = \cot t$

m $x = e^t - 5,\ y = \ln t$

$\frac{dx}{dt} = e^t,\ \frac{dy}{dt} = \frac{1}{t}$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1}{te^t}$

n $x = \ln t,\ y = t^2 - 64$

$\frac{dx}{dt} = \frac{1}{t},\ \frac{dy}{dt} = 2t$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{\frac{1}{t}} = 2t^2$

o $x = e^{2t} + 1,\ y = 2e^t - 1$

$\frac{dx}{dt} = 2e^{2t},\ \frac{dy}{dt} = 2e^t$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2e^t}{2e^{2t}} = \frac{1}{e^t} = e^{-t}$

2 a $x = 3 - 2\sin t,\ y = t\cos t$

$\frac{dx}{dt} = -2\cos t,\ \frac{dy}{dt} = -t\sin t + \cos t$

$\therefore \frac{dy}{dx} = \frac{-t\sin t + \cos t}{-2\cos t} = \frac{t}{2}\tan t - \frac{1}{2}$

At point P, where $t = \pi$,

$x = 3,\ y = -\pi$ and $\frac{dy}{dx} = -\frac{1}{2}$

Equation of tangent is

$y - (-\pi) = -\frac{1}{2}(x - 3)$

$y = -\frac{1}{2}x + \frac{3}{2} - \pi$

b $x = 9 - t^2,\ y = t^2 + 6t$

$\frac{dx}{dt} = -2t,\ \frac{dy}{dt} = 2t + 6$

$\therefore \frac{dy}{dx} = -\frac{2t + 6}{2t} = -\frac{t + 3}{t}$

At point P, where $t = 2$,

$x = 5,\ y = 16$ and $\frac{dy}{dx} = -\frac{5}{2}$

Equation of tangent is

$y - 16 = -\frac{5}{2}(x - 5)$

$2y - 32 = 25 - 5x$

$2y + 5x = 57$

3 a $x = e^t,\ y = e^t + e^{-t}$

$\frac{dx}{dt} = e^t,\ \frac{dy}{dt} = e^t - e^{-t}$

$\therefore \frac{dy}{dx} = \frac{e^t - e^{-t}}{e^t} = 1 - e^{-2t}$

At point P, where $t = 0$,

$\frac{dy}{dx} = 1 - 1 = 0$

Gradient of curve is 0

$\therefore$ normal is parallel to the y-axis.

When $t = 0,\ x = 1$ and $y = 2$

Equation of the normal is $x = 1$.

3 b $x = 1 - \cos 2t,\ y = \sin 2t$

$$\frac{dx}{dt} = 2\sin 2t,\ \frac{dy}{dt} = 2\cos 2t$$

$$\therefore \frac{dy}{dx} = \frac{2\cos 2t}{2\sin 2t} = \cot 2t$$

At point P, where $t = \frac{\pi}{6}$,

$$\frac{dy}{dx} = \frac{1}{\tan\frac{\pi}{3}} = \frac{1}{\sqrt{3}}$$

$\therefore$ gradient of the normal is $-\sqrt{3}$

When $t = \frac{\pi}{6},\ x = 1 - \cos\frac{\pi}{3} = \frac{1}{2}$

and $y = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

Equation of the normal is

$$y - \frac{\sqrt{3}}{2} = -\sqrt{3}\left(x - \frac{1}{2}\right)$$

$$y - \frac{\sqrt{3}}{2} = -\sqrt{3}x + \frac{\sqrt{3}}{2}$$

$$y + \sqrt{3}x = \sqrt{3}$$

4 $x = \frac{t}{1-t},\ y = \frac{t^2}{1-t}$

Using the quotient rule,

$$\frac{dx}{dt} = \frac{(1-t)\times 1 - t(-1)}{(1-t)^2} = \frac{1}{(1-t)^2}$$

$$\frac{dy}{dt} = \frac{(1-t)2t - t^2(-1)}{(1-t)^2} = \frac{2t - t^2}{(1-t)^2}$$

$$\therefore \frac{dy}{dx} = \frac{2t - t^2}{(1-t)^2} \div \frac{1}{(1-t)^2}$$

$$= 2t - t^2 = t(2-t)$$

$\frac{dy}{dx} = 0$ when $t = 0$ or 2

When $t = 0$, $x = 0$ and $y = 0$

When $t = 2$, $x = -2$ and $y = -4$

$\therefore$ (0, 0) and (−2, −4) are the points of zero gradient on the curve.

5 a $x = e^{2t},\ y = e^t - 1$

$$\frac{dx}{dt} = 2e^{2t},\ \frac{dy}{dt} = e^t$$

$$\therefore \frac{dy}{dx} = \frac{e^t}{2e^{2t}} = \frac{1}{2e^t}$$

When $t = \ln 2$,

$x = 4,\ y = 1$ and $\frac{dy}{dx} = \frac{1}{4}$

Equation of tangent is

$$y - 1 = \tfrac{1}{4}(x - 4)$$

$$y = \tfrac{1}{4}x$$

b At stationary points $\frac{dy}{dx} = 0$

$$\frac{1}{2e^t} = 0 \Rightarrow e^{-t} = 0$$

This has no solutions, so the curve has no stationary points.

6 $x = \dfrac{t^2 - 3t - 4}{t}$, $y = 2t$

$x = \dfrac{t^2}{t} - 3 - \dfrac{4}{t} = t - 3 - 4t^{-1}$

$\dfrac{dx}{dt} = 1 + \dfrac{4}{t^2}$, $\dfrac{dy}{dt} = 2$

$\therefore \dfrac{dy}{dx} = \dfrac{2}{1 + \frac{4}{t^2}} = \dfrac{2t^2}{t^2 + 4}$

l_1 is parallel to $y = x + 5$

so gradient of l_1 is 1

$\dfrac{dy}{dx} = 1 \Rightarrow \dfrac{2t^2}{t^2+4} = 1$

$\Rightarrow t^2 = 4$

so $t = 2$ (because $t > 0$)

When $t = 2$, $x = -3$ and $y = 4$

Equation of l_1 is

$y - 4 = 1 \times (x + 3)$

$y = x + 7$

7 a $x = 2\sin^2 t$, $y = 2\cot t$

$\dfrac{dx}{dt} = 4\sin t\cos t$, $\dfrac{dy}{dt} = -2\operatorname{cosec}^2 t$

$\therefore \dfrac{dy}{dx} = -\dfrac{2\operatorname{cosec}^2 t}{4\sin t\cos t} = -\dfrac{1}{2}\sec t\operatorname{cosec}^3 t$

b When $t = \dfrac{\pi}{6}$, $x = \dfrac{1}{2}$, $y = 2\sqrt{3}$

and $\dfrac{dy}{dx} = -\dfrac{1}{2} \times \dfrac{2}{\sqrt{3}} \times 2^3 = -\dfrac{8}{\sqrt{3}} = -\dfrac{8\sqrt{3}}{3}$

Equation of tangent is

$y - 2\sqrt{3} = -\dfrac{8\sqrt{3}}{3}\left(x - \dfrac{1}{2}\right)$

$\sqrt{3}y - 6 = -8x + 4$

$8x + \sqrt{3}y - 10 = 0$

8 a $x = 4\sin t$, $y = 2\operatorname{cosec} 2t$

$x = 2\sqrt{3} \Rightarrow 4\sin t = 2\sqrt{3}$

$\sin t = \dfrac{\sqrt{3}}{2}$ $\therefore t = \dfrac{\pi}{3}, \dfrac{2\pi}{3}$

$t = \dfrac{\pi}{3} \Rightarrow y = 2\operatorname{cosec}\dfrac{2\pi}{3} = \dfrac{4}{\sqrt{3}} = \dfrac{4\sqrt{3}}{3}$,

which is the y-coordinate of point A.

So $t = \dfrac{\pi}{3}$ at point A.

b $\dfrac{dx}{dt} = 4\cos t$, $\dfrac{dy}{dt} = -4\operatorname{cosec} 2t\cot 2t$

$\therefore \dfrac{dy}{dx} = -\dfrac{-4\operatorname{cosec} 2t\cot 2t}{4\cos t}$

$= -\dfrac{\cot 2t\operatorname{cosec} 2t}{\cos t}$

When $t = \dfrac{\pi}{3}$, $\dfrac{dy}{dx} = -\dfrac{\cot\frac{2\pi}{3}\operatorname{cosec}\frac{2\pi}{3}}{\cos\frac{\pi}{3}}$

$= -\dfrac{\left(-\frac{1}{\sqrt{3}}\right) \times \frac{2}{\sqrt{3}}}{\frac{1}{2}} = \dfrac{4}{3}$

$\therefore$ gradient of normal is $-\frac{3}{4}$

Equation of normal, l, is

$y - \dfrac{4\sqrt{3}}{3} = -\dfrac{3}{4}(x - 2\sqrt{3})$

$12y - 16\sqrt{3} = -9(x - 2\sqrt{3})$

$9x + 12y - 34\sqrt{3} = 0$

9 a $x = t^2 + t$, $y = t^2 - 10t + 5$

$\dfrac{dx}{dt} = 2t + 1$, $\dfrac{dy}{dt} = 2t - 10$

$\therefore \dfrac{dy}{dx} = \dfrac{2t - 10}{2t + 1}$

When gradient is 2, $\dfrac{2t - 10}{2t + 1} = 2$

$2t - 10 = 4t + 2 \Rightarrow t = -6$

At P, $x = (-6)^2 - 6 = 30$

and $y = (-6)^2 - 10(-6) + 5 = 101$

Coordinates of P are (30, 101).

9 b Equation of tangent at P is

$y - 101 = 2(x - 30)$

$y = 2x + 41$

c Substituting for y and x in the tangent equation:

$t^2 - 10t + 5 = 2(t^2 + t) + 41$

$t^2 + 12t + 36 = 0$

Discriminant $= 12^2 - 4 \times 36 = 0$

Therefore the curve and the line only intersect once, so the tangent at P does not intersect the curve again.

10 a $x = 2\sin t,\ y = \sqrt{2}\cos 2t$

$\frac{dx}{dt} = 2\cos t,\ \frac{dy}{dt} = -2\sqrt{2}\sin 2t$

$\therefore \frac{dy}{dx} = \frac{-2\sqrt{2}\sin 2t}{2\cos t} = \frac{-\sqrt{2} \times 2\sin t\cos t}{\cos t}$

$= -2\sqrt{2}\sin t$

b When $t = \frac{\pi}{3}$:

$x = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3},\ y = \sqrt{2}\left(-\frac{1}{2}\right) = -\frac{\sqrt{2}}{2}$

and $\frac{dy}{dx} = -2\sqrt{2}\left(\frac{\sqrt{3}}{2}\right) = -\sqrt{6}$

Equation of normal at A is

$y - \left(-\frac{\sqrt{2}}{2}\right) = \frac{1}{\sqrt{6}}(x - \sqrt{3})$

$\sqrt{6}y + \sqrt{3} = x - \sqrt{3}$

$x - \sqrt{6}y - 2\sqrt{3} = 0$

c Substituting for y and x in the normal equation:

$2\sin t - \sqrt{6} \times \sqrt{2}\cos 2t - 2\sqrt{3} = 0$

$\sin t - \sqrt{3}\cos 2t - \sqrt{3} = 0$

$\sin t - \sqrt{3}(1 - 2\sin^2 t) - \sqrt{3} = 0$

$2\sqrt{3}\sin^2 t + \sin t - 2\sqrt{3} = 0$

$(2\sin t - \sqrt{3})(\sqrt{3}\sin t + 2) = 0$

$\sin t = \frac{\sqrt{3}}{2}$ or $\sin t = -\frac{2}{\sqrt{3}}$

(2nd option not possible since $|\sin t| \leqslant 1$)

$\sin t = \frac{\sqrt{3}}{2} \Rightarrow t = \frac{\pi}{3}$ or $\frac{2\pi}{3}$

When $t = \frac{2\pi}{3}$:

$x = 2\sin\frac{2\pi}{3} = \sqrt{3},\ y = \sqrt{2}\cos\frac{4\pi}{3} = -\frac{\sqrt{2}}{2},$

which is the same as point A, so l does not intersect C other than at point A.

11 a $x = \cos t,\ y = \frac{1}{2}\sin 2t$

$\frac{dx}{dt} = -\sin t,\ \frac{dy}{dt} = \cos 2t$

$\therefore \frac{dy}{dx} = -\frac{\cos 2t}{\sin t}$

b When $t = \frac{\pi}{6}$: $x = \frac{\sqrt{3}}{2},\ y = \frac{\sqrt{3}}{4}$

and $\frac{dy}{dx} = -\frac{\frac{1}{2}}{\frac{1}{2}} = -1$

Equation of tangent at A is

$y - \frac{\sqrt{3}}{4} = -\left(x - \frac{\sqrt{3}}{2}\right)$

i.e. $y = -x + \frac{3\sqrt{3}}{4}$

11 c l_1 and l_2 both have gradient -1

$\therefore$ values of t at points where the tangents cut the curve will be solutions to

$$-\frac{\cos 2t}{\sin t} = -1$$

$$1 - 2\sin^2 t = \sin t$$

$$2\sin^2 t + \sin t - 1 = 0$$

$$(2\sin t - 1)(\sin t + 1) = 0$$

$$\sin t = \tfrac{1}{2} \text{ or } -1$$

$$t = \frac{\pi}{6}, \frac{5\pi}{6} \text{ or } \frac{3\pi}{2}$$

So lines l_1 and l_2 touch the curve when

$t = \dfrac{5\pi}{6}$ and $t = \dfrac{3\pi}{2}$.

$$t = \frac{5\pi}{6} \Rightarrow x = -\frac{\sqrt{3}}{2}, y = -\frac{\sqrt{3}}{4}$$

Equation of l_1 is

$$y - \left(-\frac{\sqrt{3}}{4}\right) = -1\left(x - \left(-\frac{\sqrt{3}}{2}\right)\right)$$

i.e. $y = -x - \dfrac{3\sqrt{3}}{4}$

$$t = \frac{3\pi}{2} \Rightarrow x = 0, y = 0$$

Equation of l_2 is

$y - 0 = -(x - 0)$

i.e. $y = -x$