

Exercise 4A

$$\begin{aligned} \mathbf{1a\ i} \quad (1+x)^{-4} &= 1 + (-4)x + \frac{(-4)(-5)}{2!}x^2 + \frac{(-4)(-5)(-6)}{3!}x^3 + \dots \\ &= 1 - 4x + 10x^2 - 20x^3 + \dots \end{aligned}$$

$$\mathbf{ii} \quad |x| < 1$$

$$\begin{aligned} \mathbf{b\ i} \quad (1+x)^{-6} &= 1 + (-6)x + \frac{(-6)(-7)}{2!}x^2 + \frac{(-6)(-7)(-8)}{3!}x^3 + \dots \\ &= 1 - 6x + 21x^2 - 56x^3 + \dots \end{aligned}$$

$$\mathbf{ii} \quad |x| < 1$$

$$\begin{aligned} \mathbf{c\ i} \quad (1+x)^{\frac{1}{2}} &= 1 + \left(\frac{1}{2}\right)x + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}x^2 + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}x^3 + \dots \\ &= 1 + \left(\frac{1}{2}\right)x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}x^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}x^3 + \dots \\ &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \dots \end{aligned}$$

$$\mathbf{ii} \quad |x| < 1$$

$$\begin{aligned} \mathbf{d\ i} \quad (1+x)^{\frac{5}{3}} &= 1 + \left(\frac{5}{3}\right)x + \frac{\left(\frac{5}{3}\right)\left(\frac{5}{3}-1\right)}{2!}x^2 + \frac{\left(\frac{5}{3}\right)\left(\frac{5}{3}-1\right)\left(\frac{5}{3}-2\right)}{3!}x^3 + \dots \\ &= 1 + \left(\frac{5}{3}\right)x + \frac{\left(\frac{5}{3}\right)\left(\frac{2}{3}\right)}{2}x^2 + \frac{\left(\frac{5}{3}\right)\left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)}{6}x^3 + \dots \\ &= 1 + \frac{5}{3}x + \frac{5}{9}x^2 - \frac{5}{81}x^3 + \dots \end{aligned}$$

$$\mathbf{ii} \quad |x| < 1$$

$$\begin{aligned} \mathbf{e\ i} \quad (1+x)^{-\frac{1}{4}} &= 1 + \left(-\frac{1}{4}\right)x + \frac{\left(-\frac{1}{4}\right)\left(-\frac{1}{4}-1\right)}{2!}x^2 + \frac{\left(-\frac{1}{4}\right)\left(-\frac{1}{4}-1\right)\left(-\frac{1}{4}-2\right)}{3!}x^3 + \dots \\ &= 1 + \left(-\frac{1}{4}\right)x + \frac{\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)}{2}x^2 + \frac{\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)\left(-\frac{9}{4}\right)}{6}x^3 + \dots \\ &= 1 - \frac{1}{4}x + \frac{5}{32}x^2 - \frac{15}{128}x^3 + \dots \end{aligned}$$

$$\mathbf{ii} \quad |x| < 1$$

$$\begin{aligned} \mathbf{f\ i} \quad (1+x)^{-\frac{3}{2}} &= 1 + \left(-\frac{3}{2}\right)x + \frac{\left(-\frac{3}{2}\right)\left(-\frac{3}{2}-1\right)}{2!}x^2 + \frac{\left(-\frac{3}{2}\right)\left(-\frac{3}{2}-1\right)\left(-\frac{3}{2}-2\right)}{3!}x^3 + \dots \\ &= 1 + \left(-\frac{3}{2}\right)x + \frac{\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{2}x^2 + \frac{\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{6}x^3 + \dots \\ &= 1 - \frac{3}{2}x + \frac{15}{8}x^2 - \frac{35}{16}x^3 + \dots \end{aligned}$$

$$\mathbf{ii} \quad |x| < 1$$

$$\begin{aligned}
 \mathbf{2 \ a \ i} \quad (1+3x)^{-3} &= 1 + (-3)(3x) + \frac{(-3)(-4)}{2!}(3x)^2 + \frac{(-3)(-4)(-5)}{3!}(3x)^3 + \dots \\
 &= 1 + (-3)(3x) + \frac{(-3)(-4)}{2}9x^2 + \frac{(-3)(-4)(-5)}{6}27x^3 + \dots \\
 &= 1 - 9x + 54x^2 - 270x^3 + \dots
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{ii} \quad &|3x| < 1 \\
 &|x| < \frac{1}{3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b \ i} \quad (1+\frac{1}{2}x)^{-5} &= 1 + (-5)(\frac{1}{2}x) + \frac{(-5)(-6)}{2!}(\frac{1}{2}x)^2 + \frac{(-5)(-6)(-7)}{3!}(\frac{1}{2}x)^3 + \dots \\
 &= 1 + (-5)(\frac{1}{2}x) + \frac{(-5)(-6)}{2}(\frac{1}{4}x^2) + \frac{(-5)(-6)(-7)}{6}(\frac{1}{8}x^3) + \dots \\
 &= 1 - \frac{5}{2}x + \frac{15}{4}x^2 - \frac{35}{8}x^3 + \dots
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{ii} \quad &|\frac{1}{2}x| < 1 \\
 &|x| < 2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c \ i} \quad (1+2x)^{\frac{3}{4}} &= 1 + (\frac{3}{4})(2x) + \frac{(\frac{3}{4})(\frac{3}{4}-1)}{2!}(2x)^2 + \frac{(\frac{3}{4})(\frac{3}{4}-1)(\frac{3}{4}-2)}{3!}(2x)^3 + \dots \\
 &= 1 + (\frac{3}{4})(2x) + \frac{(\frac{3}{4})(-\frac{1}{4})}{2}4x^2 + \frac{(\frac{3}{4})(-\frac{1}{4})(-\frac{5}{4})}{6}8x^3 + \dots \\
 &= 1 + \frac{3}{2}x - \frac{3}{8}x^2 + \frac{5}{16}x^3 + \dots
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{ii} \quad &|2x| < 1 \\
 &|x| < \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d \ i} \quad (1-5x)^{\frac{7}{3}} &= 1 + (\frac{7}{3})(-5x) + \frac{(\frac{7}{3})(\frac{7}{3}-1)}{2!}(-5x)^2 + \frac{(\frac{7}{3})(\frac{7}{3}-1)(\frac{7}{3}-2)}{3!}(-5x)^3 + \dots \\
 &= 1 - (\frac{7}{3})5x + \frac{(\frac{7}{3})(\frac{4}{3})}{2}25x^2 - \frac{(\frac{7}{3})(\frac{4}{3})(\frac{1}{3})}{6}125x^3 + \dots \\
 &= 1 - \frac{35}{3}x + \frac{350}{9}x^2 - \frac{1750}{81}x^3 + \dots
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{ii} \quad &|-5x| < 1 \\
 &|x| < \frac{1}{5}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e \ i} \quad (1+6x)^{-\frac{2}{3}} &= 1 + (-\frac{2}{3})(6x) + \frac{(-\frac{2}{3})(-\frac{2}{3}-1)}{2!}(6x)^2 + \frac{(-\frac{2}{3})(-\frac{2}{3}-1)(-\frac{2}{3}-2)}{3!}(6x)^3 + \dots \\
 &= 1 + (-\frac{2}{3})(6x) + \frac{(-\frac{2}{3})(-\frac{5}{3})}{2}36x^2 + \frac{(-\frac{2}{3})(-\frac{5}{3})(-\frac{8}{3})}{6}216x^3 + \dots \\
 &= 1 - 4x + 20x^2 - \frac{320}{3}x^3 + \dots
 \end{aligned}$$

2 e ii $|6x| < 1$
 $|x| < \frac{1}{6}$

f i $(1 - \frac{3}{4}x)^{-\frac{5}{3}} = 1 + (-\frac{5}{3})(-\frac{3}{4}x) + \frac{(-\frac{5}{3})(-\frac{5}{3}-1)}{2!}(-\frac{3}{4}x)^2 + \frac{(-\frac{5}{3})(-\frac{5}{3}-1)(-\frac{5}{3}-2)}{3!}(-\frac{3}{4}x)^3 + \dots$
 $= 1 + (-\frac{5}{3})(-\frac{3}{4}x) + \frac{(-\frac{5}{3})(-\frac{8}{3})}{2}(\frac{9}{16}x^2) - \frac{(-\frac{5}{3})(-\frac{8}{3})(-\frac{11}{3})}{6}(\frac{27}{64}x^3) + \dots$
 $= 1 + \frac{5}{4}x + \frac{5}{4}x^2 + \frac{55}{48}x^3 + \dots$

ii $|-\frac{3}{4}x| < 1$
 $|x| < \frac{4}{3}$

3 a i $\frac{1}{(1+x)^2} = (1+x)^{-2} = 1 + (-2)x + \frac{(-2)(-3)}{2!}x^2 + \frac{(-2)(-3)(-4)}{3!}x^3 + \dots$
 $= 1 - 2x + 3x^2 - 4x^3 + \dots$

ii $|x| < 1$

b i $\frac{1}{(1+3x)^4} = (1+3x)^{-4} = 1 + (-4)(3x) + \frac{(-4)(-5)}{2!}(3x)^2 + \frac{(-4)(-5)(-6)}{3!}(3x)^3 + \dots$
 $= 1 + (-4)(3x) + \frac{(-4)(-5)}{2}9x^2 + \frac{(-4)(-5)(-6)}{6}27x^3 + \dots$
 $= 1 - 12x + 90x^2 - 540x^3 + \dots$

ii $|3x| < 1$
 $|x| < \frac{1}{3}$

c i $\sqrt{1-x} = (1-x)^{\frac{1}{2}} = 1 + (\frac{1}{2})(-x) + \frac{(\frac{1}{2})(\frac{1}{2}-1)}{2!}(-x)^2 + \frac{(\frac{1}{2})(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}(-x)^3 + \dots$
 $= 1 - (\frac{1}{2})x + \frac{(\frac{1}{2})(-\frac{1}{2})}{2}x^2 - \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{6}x^3 + \dots$
 $= 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 + \dots$

ii $|-x| < 1$
 $|x| < 1$

d i $\sqrt[3]{1-3x} = (1-3x)^{\frac{1}{3}} = 1 + (\frac{1}{3})(-3x) + \frac{(\frac{1}{3})(\frac{1}{3}-1)}{2!}(-3x)^2 + \frac{(\frac{1}{3})(\frac{1}{3}-1)(\frac{1}{3}-2)}{3!}(-3x)^3 + \dots$
 $= 1 - (\frac{1}{3})3x + \frac{(\frac{1}{3})(-\frac{2}{3})}{2}9x^2 - \frac{(\frac{1}{3})(-\frac{2}{3})(-\frac{5}{3})}{6}27x^3 + \dots$
 $= 1 - x - x^2 - \frac{5}{3}x^3 + \dots$

3 d ii $|-3x| < 1$
 $|x| < \frac{1}{3}$

e i
$$\frac{1}{\sqrt{1+\frac{1}{2}x}} = (1+\frac{1}{2}x)^{-\frac{1}{2}} = 1 + (-\frac{1}{2})(\frac{1}{2}x) + \frac{(-\frac{1}{2})(-\frac{1}{2}-1)}{2!}(\frac{1}{2}x)^2 + \frac{(-\frac{1}{2})(-\frac{1}{2}-1)(-\frac{1}{2}-2)}{3!}(\frac{1}{2}x)^3 + \dots$$

$$= 1 + (-\frac{1}{2})(\frac{1}{2}x) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2}(\frac{1}{4}x^2) + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{6}(\frac{1}{8}x^3) + \dots$$

$$= 1 - \frac{1}{4}x + \frac{3}{32}x^2 - \frac{5}{128}x^3 + \dots$$

ii $|\frac{1}{2}x| < 1$
 $|x| < 2$

f i
$$\frac{\sqrt[3]{1-2x}}{1-2x} = (1-2x)^{-\frac{2}{3}} = 1 + (-\frac{2}{3})(-2x) + \frac{(-\frac{2}{3})(-\frac{2}{3}-1)}{2!}(-2x)^2 + \frac{(-\frac{2}{3})(-\frac{2}{3}-1)(-\frac{2}{3}-2)}{3!}(-2x)^3 + \dots$$

$$= 1 - (-\frac{2}{3})2x + \frac{(-\frac{2}{3})(-\frac{5}{3})}{2}4x^2 - \frac{(-\frac{2}{3})(-\frac{5}{3})(-\frac{8}{3})}{6}8x^3 + \dots$$

$$= 1 + \frac{4}{3}x + \frac{20}{9}x^2 + \frac{320}{81}x^3 + \dots$$

ii $|-2x| < 1$
 $|x| < \frac{1}{2}$

4 a
$$\frac{1+x}{1-2x} = (1+x)(1-2x)^{-1}$$
 Expand $(1-2x)^{-1}$ using binomial expansion
$$= (1+x) \left(1 + (-1)(-2x) + \frac{(-1)(-2)(-2x)^2}{2!} + \frac{(-1)(-2)(-3)(-2x)^3}{3!} + \dots \right)$$

$$= (1+x)(1+2x+4x^2+8x^3+\dots)$$
 Multiply out
$$= 1+2x+4x^2+8x^3+\dots+x+2x^2+4x^3+8x^4+\dots$$
 Add like terms
$$= 1+3x+6x^2+12x^3+\dots$$

b $(1-2x)^{-1}$ is only valid when $|-2x| < 1 \Rightarrow |x| < \frac{1}{2}$

So expansion of $\frac{1+x}{1-2x}$ is only valid when $|x| < \frac{1}{2}$

5 a
$$f(x) = (1+3x)^{\frac{1}{2}} = 1 + (\frac{1}{2})(3x) + \frac{(\frac{1}{2})(\frac{1}{2}-1)}{2!}(3x)^2 + \frac{(\frac{1}{2})(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}(3x)^3 + \dots$$

$$= 1 + (\frac{1}{2})(3x) + \frac{(\frac{1}{2})(-\frac{1}{2})}{2}9x^2 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{6}27x^3 + \dots$$

$$= 1 + \frac{3x}{2} - \frac{9x^2}{8} + \frac{27x^3}{16} + \dots$$

b When $x = \frac{1}{100}$, $f(x) = \sqrt{1+3\left(\frac{1}{100}\right)} = \sqrt{\frac{103}{100}} = \frac{\sqrt{103}}{10}$

5 c Using the expansion:

$$f(0.01) \approx 1 + \frac{3(0.01)}{2} - \frac{9(0.01)^2}{8} + \frac{27(0.01)^3}{16}$$

$$= 1.014889188\dots$$

$$\text{Percentage error} = \frac{1.014889188 - \frac{\sqrt{103}}{10}}{\frac{\sqrt{103}}{10}} \times 100 = 0.0000031\%$$

6 a $(1+ax)^{-\frac{1}{2}} = 1 + (-\frac{1}{2})(ax) + \frac{(-\frac{1}{2})(-\frac{1}{2}-1)}{2!}(ax)^2 + \frac{(-\frac{1}{2})(-\frac{1}{2}-1)(-\frac{1}{2}-2)}{3!}(ax)^3 + \dots$

$$= 1 + (-\frac{1}{2})(ax) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2}a^2x^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{6}a^3x^3 + \dots$$

$$= 1 - \frac{ax}{2} + \frac{3a^2x^2}{8} - \frac{5a^3x^3}{16} + \dots$$

$$\frac{3a^2}{8} = 24$$

$$a^2 = 64$$

$$a = \pm 8$$

b When $a = 8$, $-\frac{5(8)^3}{16} = -160$

When $a = -8$, $-\frac{5(-8)^3}{16} = 160$

7 For small values of x , ignore terms in x^3 and higher.

$$\sqrt{\frac{1+x}{1-x}} = (1+x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$$

$$(1+x)^{\frac{1}{2}} = 1 + \left(\frac{1}{2}\right)x + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}x^2 + \dots$$

$$= 1 + \left(\frac{1}{2}\right)x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}x^2 + \dots$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \dots$$

$$(1-x)^{-\frac{1}{2}} = 1 + \left(-\frac{1}{2}\right)(-x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}(-x)^2 + \dots$$

$$= 1 + \frac{1}{2}x + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}x^2 + \dots$$

$$= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \dots$$

$$\sqrt{\frac{1+x}{1-x}} = (1+x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}} = \left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \dots\right)\left(1 + \frac{1}{2}x + \frac{3}{8}x^2 + \dots\right)$$

$$= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{1}{2}x + \frac{1}{4}x^2 - \frac{1}{8}x^2 + \dots$$

$$= 1 + x + \frac{1}{2}x^2 + \dots$$

$$8 \text{ a } h(x) = \frac{6}{1+5x} - \frac{4}{1-3x} = 6(1+5x)^{-1} - 4(1-3x)^{-1}$$

$$(1+5x)^{-1} = 1 + (-1)(5x) + \frac{(-1)(-2)}{2!}(5x)^2 + \dots$$

$$= 1 + (-1)(5x) + \frac{(-1)(-2)}{2}25x^2 + \dots$$

$$= 1 - 5x + 25x^2 + \dots$$

$$(1-3x)^{-1} = 1 + (-1)(-3x) + \frac{(-1)(-2)}{2!}(-3x)^2 + \dots$$

$$= 1 + (-1)(-3x) + \frac{(-1)(-2)}{2}9x^2 + \dots$$

$$= 1 + 3x + 9x^2 + \dots$$

$$h(x) = 6(1+5x)^{-1} - 4(1-3x)^{-1}$$

$$= 6(1 - 5x + 25x^2 + \dots) - 4(1 + 3x + 9x^2 + \dots)$$

$$= 6 - 30x + 150x^2 - 4 - 12x - 36x^2 + \dots$$

$$= 2 - 42x + 114x^2 + \dots$$

$$b \quad h(0.01) = \frac{6}{1+5(0.01)} - \frac{4}{1-3(0.01)} = 1.590574374$$

$$h(0.01) = 2 - 42(0.01) + 114(0.01)^2 = 1.5914$$

$$\frac{1.5914 - 1.590574374}{1.590574374} \times 100 = 0.052\%$$

c The expansion is only valid for $|x| < \frac{1}{5}$. $|0.5|$ is not less than $\frac{1}{5}$

$$9 \text{ a } (1-3x)^{\frac{3}{2}} = 1 + \left(\frac{3}{2}\right)(-3x) + \frac{\left(\frac{3}{2}\right)\left(\frac{3}{2}-1\right)}{2!}(-3x)^2 + \frac{\left(\frac{3}{2}\right)\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-2\right)}{3!}(-3x)^3 + \dots$$

$$= 1 + \left(\frac{3}{2}\right)(-3x) + \frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)}{2}9x^2 - \frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{6}27x^3 + \dots$$

$$= 1 - \frac{9x}{2} + \frac{27x^2}{8} + \frac{27x^3}{16} + \dots$$

$$b \text{ When } x = \frac{1}{100}, \left(1 - 3\left(\frac{1}{100}\right)\right)^{\frac{3}{2}} = \left(\frac{97}{100}\right)^{\frac{3}{2}} = \left(\frac{\sqrt{97}}{10}\right)^3 = \frac{97\sqrt{97}}{1000}$$

$$c \quad (\sqrt{0.97})^3 = 1 - \frac{9(0.01)}{2} + \frac{27(0.01)^2}{8} + \frac{27(0.01)^3}{16} = 0.955339\dots$$

$$\sqrt{0.97} = \sqrt[3]{0.955339\dots} = 0.984886$$

$$\sqrt{97} = \sqrt{0.97 \times 100} = 0.984886 \times 10 = 9.84886$$

Challenge

$$\begin{aligned}
 \text{a } \left(1 + \frac{1}{x}\right)^{-\frac{1}{2}} &= 1 + \left(-\frac{1}{2}\right)\left(\frac{1}{x}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(\frac{1}{x}\right)^2}{2!} + \dots \\
 &= 1 + \left(-\frac{1}{2}\right)\left(\frac{1}{x}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{1}{x^2}}{2} + \dots \\
 &= 1 - \frac{1}{2x} + \frac{3}{8x^2} + \dots
 \end{aligned}$$

$$\text{b } h(9) = \left(1 + \frac{1}{9}\right)^{-\frac{1}{2}} = \left(\frac{10}{9}\right)^{-\frac{1}{2}} = \left(\frac{9}{10}\right)^{\frac{1}{2}} = \frac{3}{\sqrt{10}} = \frac{3\sqrt{10}}{10}$$

$$\text{c } h(9) = \frac{3\sqrt{10}}{10}$$

$$\frac{10}{3}h(9) = \sqrt{10}$$

$$\text{So } \sqrt{10} = \frac{10}{3}\left(1 - \frac{1}{2(9)} + \frac{3}{8(9)^2}\right)$$

$$= \frac{10}{3}\left(1 - \frac{1}{18} + \frac{3}{648}\right)$$

$$= \frac{256}{81} = 3.16$$