Chapter review

1 a At A, $y = 0 \Rightarrow 3\sin t = 0 \Rightarrow \sin t = 0$

So $t = 0$ or $t = \pi$

Substitute $t = 0$ and $t = \pi$ into

$x = 4\cos t$

$t = 0 \Rightarrow x = 4\cos 0 = 4 \times 1 = 4$

$t = \pi \Rightarrow x = 4\cos \pi = 4 \times (-1) = -4$

The coordinates of A are $(4, 0)$.

At B, $x = 0 \Rightarrow 4\cos t = 0 \Rightarrow \cos t = 0$

So $t = \frac{\pi}{2}$ or $t = \frac{3\pi}{2}$

Substitute $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$ into

$y = 3\sin t$

$t = \frac{\pi}{2} \Rightarrow y = 3\sin\frac{\pi}{2} = 3 \times 1 = 3$

$t = \frac{3\pi}{2} \Rightarrow y = 3\sin\frac{3\pi}{2} = 3 \times -1 = -3$

The coordinates of B are $(0, 3)$.

b At C, $t = \frac{\pi}{6}$

$x = 4\cos\frac{\pi}{6} = 4 \times \frac{\sqrt{3}}{2} = 2\sqrt{3}$

$y = 3\sin\frac{\pi}{6} = 3 \times \frac{1}{2} = \frac{3}{2}$

The coordinates of C are $\left(2\sqrt{3}, \frac{3}{2}\right)$.

c $x = 4\cos t \Rightarrow \frac{x}{4} = \cos t$ (1)

$y = 3\sin t \Rightarrow \frac{y}{3} = \sin t$ (2)

Substitute (1) and (2) into

$\cos^2 t + \sin^2 t = 1$

$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$

$\frac{x^2}{16} + \frac{y^2}{9} = 1$

$9x^2 + 16y^2 = 144$

2 Substitute $t = 0$ into

$x = \cos t$ and $y = \frac{1}{2}\sin 2t$

$x = \cos 0 = 1$

$y = \frac{1}{2}\sin(2 \times 0) = \frac{1}{2}\sin 0 = \frac{1}{2} \times 0 = 0$

So when $t = 0$, $(x, y) = (1, 0)$.

Substitute $t = \frac{\pi}{2}$ into

$x = \cos t$ and $y = \frac{1}{2}\sin 2t$

$x = \cos\frac{\pi}{2} = 0$

$y = \frac{1}{2}\sin\left(2 \times \frac{\pi}{2}\right) = \frac{1}{2}\sin \pi = \frac{1}{2} \times 0 = 0$

So when $t = \frac{\pi}{2}$, $(x, y) = (0, 0)$.

Substitute $t = \pi$ into

$x = \cos t$ and $y = \frac{1}{2}\sin 2t$

$x = \cos \pi = -1$

$y = \frac{1}{2}\sin 2\pi = \frac{1}{2} \times 0 = 0$

So when $t = \pi$, $(x, y) = (-1, 0)$.

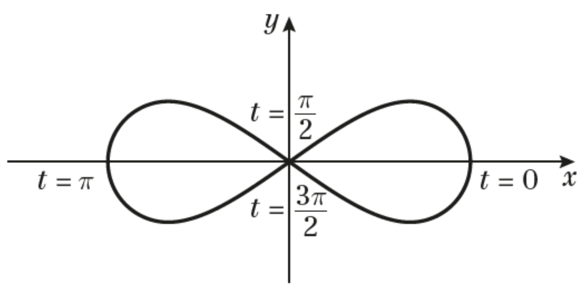
Substitute $t = 3\pi$ into

$x = \cos t$ and $y = \frac{1}{2}\sin 2t$

$x = \cos\frac{3\pi}{2} = 0$

$y = \frac{1}{2}\sin\left(2 \times \frac{3\pi}{2}\right) = \frac{1}{2}\sin 3\pi = \frac{1}{2} \times 0 = 0$

So when $t = \frac{3\pi}{2}$, $(x, y) = (0, 0)$.



3 a $x = e^{2t+1} + 1$ (1)

$x - 1 = e^{2t+1}$

$\ln(x-1) = 2t + 1$

$\ln(x-1) - 1 = 2t$

$\frac{1}{2}\ln(x-1) - \frac{1}{2} = t$ (2)

$y = t + \ln 2$ (3)

Substitute (2) into (3).

$y = \frac{1}{2}\ln(x-1) - \frac{1}{2} + \ln 2$

$= \ln(x-1)^{\frac{1}{2}} + \ln 2 - \frac{1}{2}$

$= \ln(2\sqrt{x-1}) - \frac{1}{2}$

The Cartesian equation of the curve is

$y = \ln(2\sqrt{x-1}) - \frac{1}{2}$

Substitute $t = 1$ into (1).

$x = e^3 + 1$

Since x is an increasing function and $t > 1$,

$x > e^3 + 1$

So $k = e^3 + 1$

b The range of f(x) is the range of y = q(t) so substitute $t = 1$ into (3).

$y = 1 + \ln 2$

Since y is an increasing function and $t > 1$,

$y > 1 + \ln 2$

The range of f(x) is $y > 1 + \ln 2$.

4 $x = \frac{1}{2t+1}$ (1)

$2t + 1 = \frac{1}{x}$

$2t = \frac{1}{x} - 1$

$t = \frac{1}{2x} - \frac{1}{2}$ (2)

$y = 2\ln\left(t + \frac{1}{2}\right)$ (3)

Substitute (2) into (3).

$y = 2\ln\left(\frac{1}{2x} - \frac{1}{2} + \frac{1}{2}\right)$

$= 2\ln\left(\frac{1}{2x}\right)$

$= -2\ln(2x)$

$= -2(\ln 2 + \ln x)$

$= -\ln 4 - 2\ln x$

The Cartesian equation of the curve is

$y = -\ln 4 - 2\ln x$

The domain of f(x) is the range of x = p(t) so substitute $t = \frac{1}{2}$ into (1).

$x = \frac{1}{2\left(\frac{1}{2}\right) + 1} = \frac{1}{2}$

As $t \to \infty$, $x \to 0$.

So the domain is $0 < x < \frac{1}{2}$.

The range of f(x) is the range of y = q(t) so substitute $t = \frac{1}{2}$ into (3).

$y = 2\ln\left(\frac{1}{2} + \frac{1}{2}\right) = 0$

As $t \to \infty$, $y \to \infty$

So the range is $y > 0$.

5 a $x = 4\sin t - 3 \Rightarrow \sin t = \dfrac{x+3}{4}$ (1)

$y = 4\cos t + 5 \Rightarrow \cos t = \dfrac{y-5}{4}$ (2)

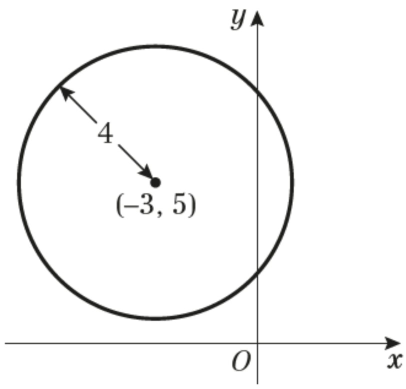
Substitute (1) and (2) into
$\sin^2 t + \cos^2 t = 1$

$$\left(\frac{x+3}{4}\right)^2 + \left(\frac{y-5}{4}\right)^2 = 1$$

$$\frac{(x+3)^2}{16} + \frac{(y-5)^2}{16} = 1$$

$$(x+3)^2 + (y-5)^2 = 16$$

b The circle $(x+3)^2 + (y-5)^2 = 4^2$ has centre $(-3, 5)$ and radius 4.



c Substitute $x = 0$ into

$$(x+3)^2 + (y-5)^2 = 16$$
$$(0+3)^2 + (y-5)^2 = 16$$
$$3^2 + (y-5)^2 = 16$$
$$9 + (y-5)^2 = 16$$
$$(y-5)^2 = 7$$
$$y - 5 = \pm\sqrt{7}$$
$$y = 5 \pm \sqrt{7}$$

The points of intersection of the circle and the y-axis are at $\left(0, 5+\sqrt{7}\right)$ and $\left(0, 5-\sqrt{7}\right)$.

6 a $x = \dfrac{2-3t}{1+t}$ (1)

$$x + xt = 2 - 3t$$
$$xt + 3t = 2 - x$$
$$t(x+3) = 2 - x$$

$t = \dfrac{2-x}{x+3}$ (2)

$y = \dfrac{3+2t}{1+t}$ (3)

Substitute (2) into (3).

$$y = \frac{3 + 2\left(\frac{2-x}{x+3}\right)}{1 + \left(\frac{2-x}{x+3}\right)}$$

$$= \frac{\frac{3(x+3) + 2(2-x)}{x+3}}{\frac{x+3+2-x}{x+3}}$$

$$= \frac{3(x+3) + 2(2-x)}{x+3+2-x}$$

$$= \frac{3x + 9 + 4 - 2x}{5}$$

$$= \frac{x+13}{5}$$

$$= \tfrac{1}{5}x + \tfrac{13}{5}$$

This is in the form $y = mx + c$, therefore the curve C is a straight line.

6 b Substitute $t = 0$ into (1) and (2).

$x = \dfrac{2-3(0)}{1+0} = 2$

$y = \dfrac{3+2(0)}{1+0} = 3$

Coordinates are $(2, 3)$.

Substitute $t = 4$ into (1) and (2).

$x = \dfrac{2-3(4)}{1+4} = -2$

$y = \dfrac{3+2(4)}{1+4} = \dfrac{11}{5}$

Coordinates are $\left(-2, \frac{11}{5}\right)$

$\text{Length} = \sqrt{(2-(-2))^2 + \left(3-\frac{11}{5}\right)^2}$

$\sqrt{(4)^2 + \left(\frac{4}{5}\right)^2}$

$= \sqrt{\dfrac{416}{25}}$

$= \dfrac{4\sqrt{26}}{5}$

7 a $x = t^2 - 2$

$x + 2 = t^2$

$\pm\sqrt{x+2} = t$

But $0 \leqslant t \leqslant 2$ so choose the positive value.

$t = \sqrt{x+2}$ (1)

Substitute (1) into

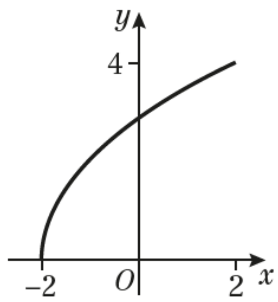
$y = 2t$

$y = 2\sqrt{x+2}$

b Domain of f(x) is $-2 \leqslant x \leqslant 2$.

Range of f(x) is $0 \leqslant y \leqslant 4$.

c



8 a $x = 2\cos t \Rightarrow \dfrac{x}{2} = \cos t$ (1)

$y = 2\sin t - 5 \Rightarrow \dfrac{y+5}{2} = \sin t$ (2)

Substitute (1) and (2) into

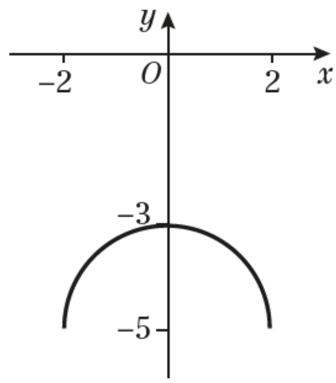
$\cos^2 t + \sin^2 t = 1$

$\left(\dfrac{x}{2}\right)^2 + \left(\dfrac{y+5}{2}\right)^2 = 1$

$x^2 + (y+5)^2 = 4$

So the curve C forms part of a circle of radius 2 and centre (0, −5).

b



c Since $0 \leqslant t \leqslant \pi$, the curve C forms half of the circle.

Arc length = $r\theta = 2\pi$

9 a $x = t - 2 \Rightarrow x + 2 = t$ (1)

Substitute (1) into

$y = t^3 - 2t^2$

$y = (x+2)^3 - 2(x+2)^2$

$= (x+2)^2(x+2-2)$

$= x(x^2 + 4x + 4)$

$= x^3 + 4x^2 + 4x$

The Cartesian equation of C is

$y = x^3 + 4x^2 + 4x$

b