Chapter review 2

1 $\dfrac{4}{(x+1)(x-4)}=\dfrac{A}{x+1}+\dfrac{B}{x-4}$

$4=A(x-4)+B(x+1)$

$4=(A+B)x+(B-4A)$

Comparing coefficients
For x:
$A+B=0$ **(1)**
For constant term:
$B-4A=4$ **(2)**
Subtracting **(1)** from **(2)** gives:
$B-4A-A-B=4-0$
$-5A=4$

$A=-\dfrac{4}{5}$

Substituting $A=-\dfrac{4}{5}$ into **(1)** gives:

$\left(-\dfrac{4}{5}\right)+B=0$

$B=\dfrac{4}{5}$

Therefore:

$\dfrac{4}{(x+1)(x-4)}=-\dfrac{4}{5(x+1)}+\dfrac{4}{5(x-4)}$

2 a $\dfrac{8x+13}{(x+2)(x+1)}=\dfrac{A}{x+2}+\dfrac{B}{x+1}$

$8x+13=A(x+1)+B(x+2)$

$=(A+B)x+(A+2B)$

Comparing coefficients
For x:
$A+B=8$ **(1)**
For constant term:
$A+2B=13$ **(2)**
Subtracting **(1)** from **(2)** gives:
$A+2B-A-B=13-8$
$B=5$
Substituting $B=5$ into **(1)** gives:
$A+(5)=8$
$A=3$
Therefore:

$\dfrac{8x+13}{(x+2)(x+1)}=\dfrac{3}{x+2}+\dfrac{5}{x+1}$

2 **b** $\frac{3-x}{(x-1)(x+3)} \equiv \frac{A}{x-1} + \frac{B}{x+3}$

$3 - x \equiv A(x+3) + B(x-1)$

$\equiv (A+B)x + (3A-B)$

Comparing coefficients

For x:

$A + B = -1$ **(1)**

For constant term:

$3A - B = 3$ **(2)**

Adding **(1)** and **(2)** gives:

$A + B + 3A - B = -1 + 3$

$4A = 2$

$A = \frac{1}{2}$

Substituting $A = \frac{1}{2}$ into **(1)** gives:

$\frac{1}{2} + B = -1$

$B = -\frac{3}{2}$

Therefore:

$\frac{3-x}{(x-1)(x+3)} \equiv \frac{1}{2(x-1)} - \frac{3}{2(x+3)}$

3 $\frac{x}{(x+1)(x-2)(x+5)} \equiv \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{x+5}$

$\frac{x}{(x+1)(x-2)(x+5)} \equiv A(x-2)(x+5) + B(x+1)(x+5) + C(x+1)(x-2)$

$\equiv Ax^2 + 3Ax - 10A + Bx^2 + 6Bx + 5B + Cx^2 - Cx - 2C$

$\equiv (A+B+C)x^2 + (3A+6B-C)x + (5B-10A-2C)$

Comparing coefficients

For x^2:

$A + B + C = 0$ **(1)**

For x:

$3A + 6B - C = 1$ **(2)**

For constant term:

$5B - 10A - 2C = 0$ **(3)**

Adding **(1)** and **(2)** gives:

$A + B + C + 3A + 6B - C = 0 + 1$

$4A + 7B = 1$ **(4)**

Adding 2 × **(1)** and **(3)** gives:

$2A + 2B + 2C + 5B - 10A - 2C = 0 + 0$

$-8A + 7B = 0$ **(5)**

Subtracting **(5)** from **(4)** gives:
$4A + 7B + 8A - 7B = 1 - 0$
$12A = 1$
$A = \frac{1}{12}$
Substituting $A = \frac{1}{12}$ into **(4)** gives:
$4\left(\frac{1}{12}\right) + 7B = 1$
$B = \frac{2}{21}$
Substituting $A = \frac{1}{12}$ and $B = \frac{2}{21}$ into **(1)** gives:
$\frac{1}{12} + \frac{2}{21} + C = 0$
$C = -\frac{5}{28}$
Therefore:
$$\frac{x}{(x+1)(x-2)(x+5)} = \frac{1}{12(x+1)} + \frac{2}{21(x-2)} - \frac{5}{28(x+5)}$$

4 a $\frac{3x^2+7x-2}{x(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}$

$$\begin{aligned} 3x^2+7x-2 &= A(x+1)(x-1) + Bx(x-1) + Cx(x+1) \\ &= Ax^2 - A + Bx^2 - Bx + Cx^2 + Cx \\ &= (A+B+C)x^2 + (C-B)x - A \end{aligned}$$

Comparing coefficients
For constant term:
$A = 2$
For x^2:
$A + B + C = 3$
$\Rightarrow B + C = 1$ **(1)**
For x:
$C - B = 7$ **(2)**
Adding **(1)** and **(2)** gives:
$B + C + C - B = 1 + 7$
$2C = 8$
$C = 4$
Substituting $C = 4$ into **(1)** gives:
$B + (4) = 1$
$B = -3$
Therefore:
$$\frac{3x^2+7x-2}{x(x+1)(x-1)} = \frac{2}{x} - \frac{3}{x+1} + \frac{4}{x-1}$$

4 b $\frac{6x^2-7x-18}{(x^2-4)(x-3)}=\frac{6x^2-7x-18}{(x+2)(x-2)(x-3)}=\frac{A}{x+2}+\frac{B}{x-2}+\frac{C}{x-3}$

$6x^2-7x-18=A(x-2)(x-3)+B(x+2)(x-3)+C(x+2)(x-2)$

$=Ax^2-5Ax+6A+Bx^2-Bx-6B+Cx^2-4C$

$=(A+B+C)x^2+(-5A-B)x+(6A-6B-4C)$

Comparing coefficients

For x^2:

$A+B+C=6$ **(1)**

For x:

$-5A-B=-7$ **(2)**

For constant term:

$6A-6B-4C=-18$ **(3)**

Adding 4 × **(1)** and **(3)** gives:

$4A+4B+4C+6A-6B-4C=24-18$

$10A-2B=6$ **(4)**

Subtracting 2 × **(2)** from **(4)** gives:

$10A-2B-2(-5A-B)=6-2(-7)$

$20A=20$

$A=1$

Substituting $A=1$ into **(4)** gives:

$10(1)-2B=6$

$B=2$

Substituting $A=1$ and $B=2$ into **(1)** gives:

$(1)+(2)+C=6$

$C=3$

$\frac{6x^2-7x-18}{(x^2-4)(x-3)}=\frac{1}{x+2}+\frac{2}{x-2}+\frac{3}{x-3}$

4 c $\frac{x^2}{(x+1)(x+2)(x+3)} = \frac{A}{(x+1)} + \frac{B}{(x+2)} + \frac{C}{(x+3)}$

$x^2 = A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)$

$= Ax^2 + 5Ax + 6A + Bx^2 + 4Bx + 3B + Cx^2 + 3Cx + 2C$

$= (A+B+C)x^2 + (5A+4B+3C)x + (6A+3B+2C)$

Comparing coefficients

For x^2:

$A + B + C = 1$ **(1)**

For x:

$5A + 4B + 3C = 0$ **(2)**

For constant term:

$6A + 3B + 2C = 0$ **(3)**

Subtracting 3 × **(1)** from **(2)** gives:

$5A + 4B + 3C - 3A - 3B - 3C = 0 - 3$

$2A + B = -3$ **(4)**

Subtracting 2 × **(1)** from **(3)** gives:

$6A + 3B + 2C - 2A - 2B - 2C = 0 - 2$

$4A + B = -2$ **(5)**

Subtracting **(4)** from **(5)** gives:

$4A + B - 2A - B = -2 - (-3)$

$2A = 1$

$A = \frac{1}{2}$

Substituting $A = \frac{1}{2}$ into **(4)** gives:

$2\left(\frac{1}{2}\right) + B = -3$

$B = -4$

Substituting $A = \frac{1}{2}$ and $B = -4$ into **(1)** gives:

$\left(\frac{1}{2}\right) + (-4) + C = 1$

$C = \frac{9}{2}$

$\frac{x^2}{(x+1)(x+2)(x+3)} = \frac{1}{2(x+1)} - \frac{4}{(x+2)} + \frac{9}{2(x+3)}$

5 Let $f(x) = x^3 + x^2 - 14x - 24$

If $(x + 2)$ is a factor then $f(-2) = 0$

$$f(-2) = (-2)^3 + (-2)^2 - 14(-2) - 24$$
$$= -8 + 4 + 28 - 24$$
$$= 0$$

Therefore $(x + 2)$ is a factor of $f(x)$

$$\begin{array}{r} x^2 - x - 12 \\ x+2 \overline{\smash{)}\, x^3 + x^2 - 14x - 24} \\ \underline{x^3 + 2x^2} \\ -x^2 - 14x \\ \underline{x^2 + 2x} \\ -12x - 24 \\ \underline{12x + 24} \\ 0 \end{array}$$

So

$$x^3 + x^2 - 14x - 24 = (x+2)(x^2 - x - 12)$$
$$= (x+2)(x+3)(x-4)$$

Therefore:

$$\frac{3x-2}{x^3 + x^2 - 14x - 24} = \frac{3x-2}{(x+2)(x+3)(x-4)}$$

$$\frac{3x-2}{(x+2)(x+3)(x-4)} = \frac{A}{x+2} + \frac{B}{x+3} + \frac{C}{x-4}$$

$$3x - 2 = A(x+3)(x-4) + B(x+2)(x-4) + C(x+2)(x+3)$$
$$= Ax^2 - Ax - 12A + Bx^2 - 2Bx - 8B + Cx^2 + 5Cx + 6C$$
$$= (A + B + C)x^2 + (5C - A - 2B)x + (6C - 12A - 8B)$$

Comparing coefficients

For x^2:

$A + B + C = 0$ **(1)**

For x:

$5C - A - 2B = 3$ **(2)**

For constant term:

$6C - 12A - 8B = -2$ **(3)**

Adding **(1)** and **(2)** gives:

$A + B + C + 5C - A - 2B = 0 + 3$

$-B + 6C = 3$ **(4)**

Adding 12 × **(1)** and **(3)** gives:

$12A + 12B + 12C + 6C - 12A - 8B = -2$

$4B + 18C = -2$ **(5)**

Adding 4 × **(4)** and **(5)** gives:

$-4B + 24C + 4B + 18C = 12 - 2$

$42C = 10$

$$C = \frac{5}{21}$$

Substituting $C = \frac{5}{21}$ into **(4)** gives:

$-B + 6C = 3$

$-B + 6\left(\frac{5}{21}\right) = 3$

$B = -\frac{11}{7}$

Substituting $B = -\frac{11}{7}$ and $C = \frac{5}{21}$ into **(1)** gives:

$A + \left(-\frac{11}{7}\right) + \left(\frac{5}{21}\right) = 0$

$A = \frac{4}{3}$

$$\frac{3x-2}{x^3+x^2-14x-24} = \frac{4}{3(x+2)} - \frac{11}{7(x+3)} + \frac{5}{21(x-4)}$$

6 a Let $f(x) = x^3 - 2x^2 - x + 2$

If $(x + 1)$ is a factor then $f(-1) = 0$

$f(-1) = (-1)^3 - 2(-1)^2 - (-1) + 2$

$= -1 - 2 + 1 + 2$

$= 0$

Therefore $(x + 1)$ is a factor of $f(x)$

$(x + 1)(ax^2 + bx + c) = x^3 - 2x^2 - x + 2$

By inspection, the other two factors must be:

$(x + 1)$ and $(x + 2)$ or $(x - 1)$ and $(x - 2)$

Since there are negative terms in $f(x)$ then the other two factors are:

$(x - 1)$ and $(x - 2)$

Multiplying to check gives:

$(x + 1)(x - 1)(x - 2) = (x + 1)(x^2 - 3x + 2)$

$= x^3 - 3x^2 + 2x + x^2 - 3x + 2$

$= x^3 - 2x^2 - 3x + 2$

Therefore:

$x^3 - 2x^2 - x + 2 = (x + 1)(x - 1)(x - 2)$

Hence:

$$\frac{2}{x^3-2x^2-x+2} = \frac{2}{(x+1)(x-1)(x-2)}$$

$$\frac{2}{(x+1)(x-1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{x-2}$$

$2 = A(x-1)(x-2) + B(x+1)(x-2) + C(x+1)(x-1)$

$= Ax^2 - 3Ax + 2A + Bx^2 - Bx - 2B + Cx^2 - C$

$= (A+B+C)x^2 + (-3A-B)x + (2A-2B-C)$

Comparing coefficients

For x^2:

$A + B + C = 0$ **(1)**

For x:

$-3A - B = 0$ **(2)**

For constant term:

$2A - 2B - C = 2$ **(3)**

Adding **(1)** and **(3)** gives:

$A + B + C + 2A - 2B - C = 0 + 2$

$3A - B = 2$ **(4)**

Subtracting **(2)** from **(4)** gives:

$3A - B + 3A + B = 2 - 0$

$6A = 2$

$A = \frac{1}{3}$

Substituting $A = \frac{1}{3}$ into **(4)** gives:

$3\left(\frac{1}{3}\right) - B = 2$

$B = -1$

Substituting $A = \frac{1}{3}$ and $B = -1$ into **(1)** gives:

$\frac{1}{3} - 1 + C = 0$

$C = \frac{2}{3}$

Therefore:

$$\frac{2}{x^3 - 2x^2 - x + 2} = \frac{1}{3(x+1)} - \frac{1}{x-1} + \frac{2}{3(x-2)}$$

6 b $\frac{3x+1}{x^3+5x^2+6x}=\frac{3x+1}{x(x+2)(x+3)}=\frac{A}{x}+\frac{B}{x+2}+\frac{C}{x+3}$

$3x+1=A(x+2)(x+3)+Bx(x+3)+Cx(x+2)$

$=Ax^2+5Ax+6A+Bx^2+3Bx+Cx^2+2Cx$

$=(A+B+C)x^2+(5A+3B+2C)x+6A$

Comparing coefficients

For constant term:

$6A = 1 \Rightarrow A=\frac{1}{6}$

For x^2:

$A + B + C = 0 \Rightarrow B+C=-\frac{1}{6}$ **(1)**

For x:

$5A + 3B + 2C = 3 \Rightarrow 3B+2C=\frac{13}{6}$ **(2)**

Subtracting 2 × **(1)** from **(2)** gives:

$3B+2C-2B-2C=\frac{13}{6}-\left(-\frac{2}{6}\right)$

$B=\frac{5}{2}$

Substituting $B=\frac{5}{2}$ into **(1)** gives:

$\left(\frac{5}{2}\right)+C=-\frac{1}{6}$

$C=-\frac{8}{3}$

Therefore:

$\frac{3x+1}{x^3+5x^2+6x}=\frac{1}{6x}+\frac{5}{2(x+2)}-\frac{8}{3(x+3)}$

7 a $\dfrac{x}{(x-3)^2(x+1)} = \dfrac{A}{(x-3)} + \dfrac{B}{(x-3)^2} + \dfrac{C}{(x+1)}$

$x = A(x-3)(x+1) + B(x+1) + C(x-3)^2$

$= Ax^2 - 2Ax - 3A + Bx + B + Cx^2 - 6Cx + 9C$

$= (A+C)x^2 + (B-2A-6C)x + (B-3A+9C)$

Comparing coefficients

For x^2:

$A + C = 0$ **(1)**

For x:

$B - 2A - 6C = 1$ **(2)**

For constant term:

$B - 3A + 9C = 0$ **(3)**

Subtracting **(3)** from **(2)** gives:

$B - 2A - 6C - B + 3A - 9C = 1 - 0$

$A - 15C = 1$ **(4)**

Subtracting **(4)** from **(1)** gives:

$A + C - A + 15C = 0 - 1$

$16C = -1$

$C = -\dfrac{1}{16}$

Substituting $C = -\dfrac{1}{16}$ into **(4)** gives:

$A - 15\left(-\dfrac{1}{16}\right) = 1$

$A = \dfrac{1}{16}$

Substituting $A = \dfrac{1}{16}$ and $C = -\dfrac{1}{16}$ into **(2)** gives:

$B - 2\left(\dfrac{1}{16}\right) - 6\left(-\dfrac{1}{16}\right) = 1$

$B = \dfrac{3}{4}$

Therefore:

$\dfrac{x}{(x-3)^2(x+1)} = \dfrac{1}{16(x-3)} + \dfrac{3}{4(x-3)^2} - \dfrac{1}{16(x+1)}$

7 b $\frac{3}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$

$3 = Ax(x+1) + B(x+1) + Cx^2$

$= Ax^2 + Ax + Bx + B + Cx^2$

$= (A+C)x^2 + (A+B)x + B$

Comparing coefficients

For constant term:

$B = 3$

For x:

$A + B = 0 \Rightarrow A = -3$

For x^2:

$A + C = 0 \Rightarrow C = 3$

Therefore:

$\frac{3}{x^2(x+1)} = -\frac{3}{x} + \frac{3}{x^2} + \frac{3}{x+1}$

8 $\frac{2x}{(x+3)^2} = \frac{A}{x+3} + \frac{B}{(x+3)^2}$

$2x = A(x+3) + B$

$= Ax + (3A + B)$

Comparing coefficients

For x:

$A = 2$

For constant term:

$3A + B = 0 \Rightarrow B = -6$

$\frac{2x}{(x+3)^2} = \frac{2}{x+3} - \frac{6}{(x+3)^2}$

9 Let $f(x) = x^3 - 3x + 2$

If $(x - 1)$ is a factor of f(x) then f(1) = 0

$f(x) = (1)^3 - 3(1) + 2 = 0$

Therefore $(x - 1)$ is a factor of f(x)

$x^3 - 3x + 2 = (x-1)(ax + bx^2 + c)$

By inspection, $ax + bx^2 + c = (x+2)(x-1)$ or $ax + bx^2 + c = (x-2)(x+1)$

$(x-1)(x+2)(x-1) = (x^2 + x - 2)(x-1) = x^3 - 3x + 2$

Therefore:

$x^3 - 3x + 2 = (x-1)^2(x+2)$

Hence:

$$\frac{4}{x^3 - 3x + 2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

$$4 = A(x-1)(x+2) + B(x+2) + C(x-1)^2$$
$$= Ax^2 + Ax - 2A + Bx + 2B + Cx^2 - 2Cx + C$$
$$= (A+C)x^2 + (A+B-2C)x + (2B - 2A + C)$$

Comparing coefficients for x^2:

$A + C = 0$ **(1)**

For x:

$A + B - 2C = 0$ **(2)**

For constant term:

$2B - 2A + C = 4$ **(3)**

Subtracting 2 × **(2)** from **(3)** gives:

$2B - 2A + C - 2A - 2B + 4C = 4 - 0$

$-4A + 5C = 4$ **(4)**

Adding 4 × **(1)** to **(4)** gives:

$4A + 4C - 4A + 5C = 0 + 4$

$9C = 4$

$C = \frac{4}{9}$

Substituting $C = \frac{4}{9}$ into **(1)** gives:

$A + \left(\frac{4}{9}\right) = 0 \Rightarrow A = -\frac{4}{9}$

Substituting $A = -\frac{4}{9}$ and $C = \frac{4}{9}$ into **(3)** gives:

$2B - 2\left(-\frac{4}{9}\right) + \frac{4}{9} = 4$

$B = \frac{4}{3}$

Therefore:

$$\frac{4}{x^3 - 3x + 2} = -\frac{4}{9(x-1)} + \frac{4}{3(x-1)^2} + \frac{4}{9(x+2)}$$

10 a $\frac{3x-1}{x+4} = A + \frac{B}{x+4}$

$3x-1 = A(x+4)+D$

$\quad = Ax+(4A+D)$

Comparing coefficients

For x:

$A = 3$

For constant term:

$4A + D = -1 \Rightarrow D = -13$

Therefore:

$\frac{3x-1}{x+4} = 3 - \frac{13}{x+4}$

b $\frac{x^2+1}{x+2} = Ax+B+\frac{C}{x+2}$

$x^2+1 = Ax(x+2)+B(x+2)+C$

$\quad = Ax^2+(2A+B)x+(2B+C)$

Comparing coefficients

For x^2:

$A = 1$

For x:

$2A + B = 0 \Rightarrow B = -2$

For constant term:

$2B + C = 1 \Rightarrow C = 5$

Therefore:

$\frac{x^2+1}{x+2} = x-2+\frac{5}{x+2}$

11 $\frac{x^2+2}{(x-2)^2} = A+\frac{B}{x-2}+\frac{C}{(x-2)^2}$

$x^2+2 = A(x-2)^2+B(x-2)+C$

$\quad = Ax^2-4Ax+4A+Bx-2B+C$

$\quad = Ax^2+(B-4A)x+(4A-2B+C)$

Comparing coefficients

For x^2:

$A = 1$

For x:

$B - 4A = 0 \Rightarrow B = 4$

For constant term:

$4A - 2B + C = 2 \Rightarrow C = 6$

Therefore:

$\frac{x^2+2}{(x-2)^2} = 1+\frac{4}{x-2}+\frac{6}{(x-2)^2}$

12 a $\frac{3-x^2}{(x+1)(x-2)} = A + \frac{B}{x+1} + \frac{C}{x-2}$

$3-x^2 = A(x+1)(x-2) + B(x-2) + C(x+1)$

$= Ax^2 - Ax - 2A + Bx - 2B + Cx + C$

$= Ax^2 + (B-A+C)x + (C-2A-2B)$

Comparing coefficients

For x^2:

$A = -1$

For x:

$B - A + C = 0 \Rightarrow B + C = -1$ **(1)**

For constant term:

$C - 2A - 2B = 3 \Rightarrow C - 2B = 1$ **(2)**

Subtracting **(2)** from **(1)** gives:

$B + C - C + 2B = -1 - 1$

$3B = -2$

$B = -\frac{2}{3}$

Substituting $B = -\frac{2}{3}$ into **(1)** gives:

$\left(-\frac{2}{3}\right) + C = -1$

$C = -\frac{1}{3}$

Therefore:

$\frac{3-x^2}{(x+1)(x-2)} = -1 - \frac{2}{3(x+1)} - \frac{1}{3(x-2)}$

b $\frac{4x^2}{x-4} = Ax + B + \frac{C}{x-4}$

$4x^2 = Ax(x-4) + B(x-4) + C$

$= Ax^2 - 4Ax + Bx - 4B + C$

$= Ax^2 + (B-4A)x + (C-4B)$

Comparing coefficients

For x^2:

$A = 4$

For x:

$B - 4A = 0 \Rightarrow B = 16$

For constant term:

$C - 4B = 0 \Rightarrow C = 64$

Therefore:

$\frac{4x^2}{x-4} = 4x + 16 + \frac{64}{x-4}$

13 $\dfrac{x^3}{(x+3)^2} = Ax + B + \dfrac{C}{x+3} + \dfrac{D}{(x+3)^2}$

$$x^3 = Ax(x+3)^2 + B(x+3)^2 + C(x+3) + D$$
$$= Ax^3 + 6Ax^2 + 9Ax + Bx^2 + 6Bx + 9B + Cx + 3C + D$$
$$= Ax^3 + (6A+B)x^2 + (9A+6B+C)x + (9B+3C+D)$$

Comparing coefficients

For x^3:

$A = 1$

For x^2:

$6A + B = 0 \Rightarrow B = -6$

For x:

$9A + 6B + C = 0 \Rightarrow C = 27$

For constant term:

$9B + 3C + D = 0 \Rightarrow D = -27$

Therefore:

$$\frac{x^3}{(x+3)^2} = x - 6 + \frac{27}{x+3} - \frac{27}{(x+3)^2}$$