Exercise 2C

1 a $\frac{x+1}{x+2} = A + \frac{B}{x+2}$

$x+1 = A(x+2)+B$

Comparing coefficients

For x:

$A = 1$

For constant term:

$2A + B = 1 \Rightarrow B = -1$

Therefore:

$\frac{x+1}{x+2} = 1 - \frac{1}{x+2}$

b $\frac{4-x}{x+1} = A + \frac{B}{x+1}$

$4-x = A(x+1)+B$

Comparing coefficients

For x:

$A = -1$

For constant term:

$A + B = 4 \Rightarrow B = 5$

Therefore:

$\frac{4-x}{x+1} = -1 + \frac{5}{x+1}$

c $\frac{x^2+1}{x-1} = \frac{(x-1)^2+2x}{x-1}$

$= x-1+\frac{2x}{x-1}$

$\frac{2x}{x-1} = A + \frac{B}{x-1}$

$2x = A(x-1)+B$

Comparing coefficients for x:

$A = 2$

For constant term:

$-A + B = 0 \Rightarrow B = 2$

Therefore:

$\frac{2x}{x-1} = 2 + \frac{2}{x-1}$

and

$x-1+\frac{2x}{x-1} = x-1+2+\frac{2}{x-1}$

$= x+1+\frac{2}{x-1}$

1 d $\frac{2x^2}{x+3}=\frac{2(x+3)^2-2(6x+9)}{x+3}$

$=2(x+3)-\frac{6(2x+3)}{x+3}$

$-\frac{6(2x+3)}{x+3}=A+\frac{B}{x+3}$

$-6(2x+3)=A(x+3)+B$

Comparing coefficients

For x:

$A=-12$

For constant term:

$3A+B=-18 \Rightarrow B=18$

Therefore:

$-\frac{6(2x+3)}{x+3}=-12+\frac{18}{x+3}$

and

$2(x+3)-\frac{6(2x+3)}{x+3}=2(x+3)-12+\frac{18}{x+3}$

$=2x-6+\frac{18}{x+3}$

2 $\frac{x^2}{(x-2)(x+3)}=A+\frac{B}{(x-2)}+\frac{C}{(x+3)}$

$x^2=A(x-2)(x+3)+B(x+3)+C(x-2)$

$x^2=A(x^2+x-6)+B(x+3)+C(x-2)$

Comparing coefficients for x^2:

$A=1$

For x:

$A+B+C=0 \Rightarrow B+C=-1$ **(1)**

For constant term:

$-6A+3B-2C=0 \Rightarrow 3B-2C=6$ **(2)**

Adding 2 × **(1)** to **(2)** gives:

$2(B+C)+3B-2C=2(-1)+6$

$5B=4 \Rightarrow B=\frac{4}{5}$

Substituting into **(1)** gives:

$\frac{4}{5}+C=-1 \Rightarrow C=-\frac{9}{5}$

Therefore:

$\frac{x^2}{(x-2)(x+3)}=1+\frac{4}{5(x-2)}-\frac{9}{5(x+3)}$

3 $\dfrac{Ax^2+Bx}{(x-1)(x+1)}=2+\dfrac{5}{2(x-1)}+\dfrac{C}{2(x+1)}$

$$Ax^2+Bx=2(x-1)(x+1)+\frac{5}{2}(x+1)+\frac{C}{2}(x-1)$$

$$2Ax^2+2Bx=4(x-1)(x+1)+5(x+1)+C(x-1)$$

$$2Ax^2+2Bx=4x^2-4+5x+5+Cx-C$$

$$=4x^2+(C+5)x+(1-C)$$

Comparing coefficients

For x^2:

$2A = 4 \Rightarrow A = 2$

For constant:

$1 - C = 0 \Rightarrow C = 1$

For x:

$2B = C + 5 \Rightarrow B = 3$

Therefore, $A = 2$, $B = 3$ and $C = 1$

4 a $\dfrac{x^2-1}{x+3}=Ax+B+\dfrac{C}{x+3}$

$$x^2-1=Ax(x+3)+B(x+3)+C$$

$$=Ax^2+3Ax+Bx+3B+C$$

$$=Ax^2+(3A+B)x+(3B+C)$$

Comparing coefficients

For x^2:

$A = 1$

For x:

$3A + B = 0 \Rightarrow B = -3$

For constant:

$3B + C = -1 \Rightarrow C = 8$

Therefore:

$$\frac{x^2-1}{x+3}=x-3+\frac{8}{x+3}$$

4 b $\frac{2x^2-2}{x(x+3)} = A + \frac{B}{x} + \frac{C}{x+3}$

$$2x^2 - 2 = Ax(x+3) + B(x+3) + Cx$$
$$= Ax^2 + 3Ax + Bx + 3B + Cx$$
$$= Ax^2 + (3A + B + C)x + 3B$$

Comparing coefficients for x^2:

$A = 2$

For constant:

$3B = -2 \Rightarrow B = -\frac{2}{3}$

For x:

$3A + B + C = 0 \Rightarrow C = -\frac{16}{3}$

Therefore:

$$\frac{2x^2-2}{x(x+3)} = 2 - \frac{2}{3x} - \frac{16}{3(x+3)}$$

c $\frac{2-3x^2}{(2x-1)(x+1)} = A + \frac{B}{2x-1} + \frac{C}{x+1}$

$$2 - 3x^2 = A(2x-1)(x+1) + B(x+1) + C(2x-1)$$
$$= 2Ax^2 + Ax - A + Bx + B + 2Cx - C$$
$$= 2Ax^2 + (A + B + 2C)x + (B - A - C)$$

Comparing coefficients for x^2:

$2A = -3 \Rightarrow A = -\frac{3}{2}$

For x:

$A + B + 2C = 0 \Rightarrow B + 2C = \frac{3}{2}$ **(1)**

For constant:

$B - A - C = 2 \Rightarrow B - C = \frac{1}{2}$ **(2)**

Subtracting **(2)** from **(1)** gives:

$$B + 2C - (B - C) = \frac{3}{2} - \frac{1}{2}$$

$$3C = 1$$

$$C = \frac{1}{3}$$

Substituting $C = \frac{1}{3}$ into **(2)** gives:

$B - \left(\frac{1}{3}\right) = \frac{1}{2}$

$B = \frac{5}{6}$

Therefore:

$$\frac{2-3x^2}{(2x-1)(x+1)} = -\frac{3}{2} + \frac{5}{6(2x-1)} + \frac{1}{3(x+1)}$$

5 $\frac{x^3}{(x+2)(x-1)} = Ax + B + \frac{C}{x+2} + \frac{D}{x-1}$

$$\begin{aligned} x^3 &= Ax(x+2)(x-1) + B(x+2)(x-1) + C(x-1) + D(x+2) \\ &= Ax(x^2+x-2) + B(x^2+x-2) + C(x-1) + D(x+2) \\ &= Ax^3 + Ax^2 - 2Ax + Bx^2 + Bx - 2B + Cx - C + Dx + 2D \\ &= Ax^3 + (A+B)x^2 + (B-2A+C+D)x + (2D-2B-C) \end{aligned}$$

Comparing coefficients

For x^3:

$A = 1$

For x^2:

$A + B = 0 \Rightarrow B = -1$

For x:

$B - 2A + C + D = 0 \Rightarrow C + D = 3$ **(1)**

For constant:

$2D - 2B - C = 0 \Rightarrow 2D - C = -2$ **(2)**

Adding **(1)** and **(2)** gives:

$C + D + 2D - C = 3 - 2$

$3D = 1$

$D = \frac{1}{3}$

Substituting $D = \frac{1}{3}$ into **(1)** gives:

$C + \left(\frac{1}{3}\right) = 3$

$C = \frac{8}{3}$

Therefore:

$$\frac{x^3}{(x+2)(x-1)} = x - 1 + \frac{8}{3(x+2)} + \frac{1}{3(x-1)}$$

6 a $\dfrac{1+x^3}{x(x+2)} = Ax + B + \dfrac{C}{x} + \dfrac{D}{x+2}$

$$1+x^3 = Ax^2(x+2) + Bx(x+2) + C(x+2) + Dx$$
$$= Ax^3 + 2Ax^2 + Bx^2 + 2Bx + Cx + 2C + Dx$$
$$= Ax^3 + (2A+B)x^2 + (2B+C+D)x + 2C$$

Comparing coefficients for x^3:
$A = 1$
For x^2:
$2A + B = 0 \Rightarrow B = -2$
For constant:
$2C = 1 \Rightarrow C = \dfrac{1}{2}$
For x:
$2B + C + D = 0 \Rightarrow D = \dfrac{7}{2}$
Therefore:
$$\frac{1+x^3}{x(x+2)} = x - 2 + \frac{1}{2x} + \frac{7}{2(x+2)}$$

b $\dfrac{x^3 - x}{(x+2)(x-2)} = Ax + B + \dfrac{C}{x+2} + \dfrac{D}{x-2}$

$$x^3 - x = Ax(x+2)(x-2) + B(x+2)(x-2) + C(x-2) + D(x+2)$$
$$= Ax^3 - 4Ax + Bx^2 - 4B + Cx - 2C + Dx + 2D$$
$$= Ax^3 + Bx^2 + (C+D-4A)x + 2(D-2B-C)$$

Comparing coefficients for x^3:
$A = 1$
For x^2:
$B = 0$
For x:
$C + D - 4A = -1 \Rightarrow C + D = 3$ **(1)**
For constant:
$D - 2B - C = 0 \Rightarrow D - C = 0$ **(2)**
Adding **(1)** and **(2)** gives:
$C + D + D - C = 3 + 0$
$2D = 3 \Rightarrow D = \dfrac{3}{2}$
Substituting $D = \dfrac{3}{2}$ into **(2)** gives:
$\left(\dfrac{3}{2}\right) - C = 0 \Rightarrow C = \dfrac{3}{2}$
Therefore:
$$\frac{x^3 - x}{(x+2)(x-2)} = x + \frac{3}{2(x+2)} + \frac{3}{2(x-2)}$$

7 $$\frac{x^2}{(x+1)^2} = A + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$x^2 = A(x+1)^2 + B(x+1) + C$$
$$= Ax^2 + 2Ax + A + Bx + B + C$$
$$= Ax^2 + (2A+B)x + (A+B+C)$$

Comparing coefficients

For x^2:

$A = 1$

For x:

$2A + B = 0 \Rightarrow B = -2$

For constant:

$A + B + C = 0 \Rightarrow C = 1$

Therefore:

$$\frac{x^2}{(x+1)^2} = 1 - \frac{2}{x+1} + \frac{1}{(x+1)^2}$$

8 **a** $$\frac{x^2+1}{(x-2)^2} = A + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$$x^2 + 1 = A(x-2)^2 + B(x-2) + C$$
$$= Ax^2 - 4Ax + 4A + Bx - 2B + C$$
$$= Ax^2 + (B-4A)x + (4A-2B+C)$$

Comparing coefficients

For x^2:

$A = 1$

For x:

$B - 4A = 0 \Rightarrow B = 4$

For constant:

$4A - 2B + C = 1 \Rightarrow C = 5$

Therefore:

$$\frac{x^2+1}{(x-2)^2} = 1 + \frac{4}{x-2} + \frac{5}{(x-2)^2}$$

8 b $\frac{2x^3-1}{(x+2)^2} = Ax+B+\frac{C}{x+2}+\frac{D}{(x+2)^2}$

$2x^3-1 = Ax(x+2)^2+B(x+2)^2+C(x+2)+D$

$= Ax^3+4Ax^2+4Ax+Bx^2+4Bx+4B+Cx+2C+D$

$= Ax^3+(4A+B)x^2+(4A+4B+C)x+(4B+2C+D)$

Comparing coefficients

For x^3:

$A = 2$

For x^2:

$4A + B = 0 \Rightarrow B = -8$

For x:

$4A + 4B + C = 0 \Rightarrow C = 24$

For constant:

$4B + 2C + D = -1 \Rightarrow D = -17$

Therefore:

$\frac{2x^3-1}{(x+2)^2} = 2x-8+\frac{24}{x+2}-\frac{17}{(x+2)^2}$

Challenge

$\frac{3x^3}{(x-1)^3} = A+\frac{B}{x-1}+\frac{C}{(x-1)^2}+\frac{D}{(x-1)^3}$

$3x^3 = A(x-1)^3+B(x-1)^2+C(x-1)+D$

$= Ax^3-3Ax^2+3Ax-A+Bx^2-2Bx+B+Cx-C+D$

$= Ax^3+(B-3A)x^2+(3A-2B+C)x+(B-A-C+D)$

Comparing coefficients

For x^3:

$A = 3$

For x^2:

$B - 3A = 0 \Rightarrow B = 9$

For x:

$3A - 2B + C = 0 \Rightarrow C = 9$

For constant:

$B - A - C + D = 0 \Rightarrow D = 3$

Therefore:

$\frac{3x^3}{(x-1)^3} = 3+\frac{9}{x-1}+\frac{9}{(x-1)^2}+\frac{3}{(x-1)^3}$