

Exercise 2A

Note that all questions in this exercise can be solved either by the method of substitution, or by equating coefficients. Questions **1b**, **1e** and **6b** have been solved by equating coefficients. All others have been solved using substitution.

$$\begin{aligned}
 \mathbf{1\ a} \quad \frac{6x-2}{(x-2)(x+3)} &\equiv \frac{A}{(x-2)} + \frac{B}{(x+3)} \\
 &\equiv \frac{A(x+3) + B(x-2)}{(x-2)(x+3)} \\
 6x-2 &\equiv A(x+3) + B(x-2)
 \end{aligned}$$

Let $x = 2$:

$$6 \times 2 - 2 = A(2+3) + B(2-2)$$

$$10 = 5A$$

$$A = 2$$

Let $x = -3$:

$$6 \times (-3) - 2 = A(-3+3) + B(-3-2)$$

$$-20 = B \times -5$$

$$B = 4$$

$$\text{Hence } \frac{6x-2}{(x-2)(x+3)} \equiv \frac{2}{(x-2)} + \frac{4}{(x+3)}$$

$$\begin{aligned}
 \mathbf{b} \quad \frac{2x+11}{(x+1)(x+4)} &\equiv \frac{A}{(x+1)} + \frac{B}{(x+4)} \\
 &\equiv \frac{A(x+4) + B(x+1)}{(x+1)(x+4)} \\
 2x+11 &\equiv A(x+4) + B(x+1) \\
 &\equiv Ax + 4A + Bx + B \\
 &\equiv (A+B)x + (4A+B)
 \end{aligned}$$

Equate coefficients of x :

$$2 = A + B \quad (1)$$

Equate constant terms:

$$11 = 4A + B \quad (2)$$

(2) - (1):

$$9 = 3A$$

$$A = 3$$

Substitute $A = 3$ in (1): $2 = 3 + B$

$$B = -1$$

$$\text{Hence } \frac{2x+11}{(x+1)(x+4)} \equiv \frac{3}{(x+1)} - \frac{1}{(x+4)}$$

$$\begin{aligned}
 1 \text{ c } \quad \frac{-7x-12}{2x(x-4)} &\equiv \frac{A}{2x} + \frac{B}{(x-4)} \\
 &\equiv \frac{A(x-4) + B \times 2x}{2x(x-4)} \\
 -7x-12 &\equiv A(x-4) + 2Bx
 \end{aligned}$$

Let $x = 4$:

$$\begin{aligned}
 -7 \times 4 - 12 &= A(4-4) + 2B \times 4 \\
 -40 &= 8B \\
 B &= -5
 \end{aligned}$$

Let $x = 0$:

$$\begin{aligned}
 -7 \times 0 - 12 &= A(0-4) + 2B \times 0 \\
 -12 &= -4A \\
 A &= 3
 \end{aligned}$$

$$\text{Hence } \frac{-7x-12}{2x(x-4)} \equiv \frac{3}{2x} - \frac{5}{(x-4)}$$

$$\begin{aligned}
 d \quad \frac{2x-13}{(2x+1)(x-3)} &\equiv \frac{A}{(2x+1)} + \frac{B}{(x-3)} \\
 &\equiv \frac{A(x-3) + B(2x+1)}{(2x+1)(x-3)} \\
 2x-13 &\equiv A(x-3) + B(2x+1)
 \end{aligned}$$

Let $x = 3$:

$$\begin{aligned}
 2 \times 3 - 13 &= A(3-3) + B(2 \times 3 + 1) \\
 -7 &= B \times 7 \\
 B &= -1
 \end{aligned}$$

Let $x = -\frac{1}{2}$:

$$\begin{aligned}
 2 \times \left(-\frac{1}{2}\right) - 13 &= A\left(-\frac{1}{2} - 3\right) + B\left(2 \times \left(-\frac{1}{2}\right) + 1\right) \\
 -14 &= A \times -3\frac{1}{2} \\
 A &= 4
 \end{aligned}$$

$$\text{Hence } \frac{2x-13}{(2x+1)(x-3)} \equiv \frac{4}{(2x+1)} - \frac{1}{(x-3)}$$

1 e First factorise the denominator:

$$\frac{6x+6}{x^2+9} \equiv \frac{6x+6}{(x+3)(x-3)}$$

$$\begin{aligned} \text{Then } \frac{6x+6}{(x+3)(x-3)} &\equiv \frac{A}{(x+3)} + \frac{B}{(x-3)} \\ &\equiv \frac{A(x-3) + B(x+3)}{(x+3)(x-3)} \\ 6x+6 &\equiv A(x-3) + B(x+3) \\ &\equiv Ax - 3A + Bx + 3B \\ &\equiv (A+B)x + (3B-3A) \end{aligned}$$

Equate coefficients of x :

$$6 = A + B \quad (1)$$

Equate constant terms:

$$6 = 3B - 3A \quad (2)$$

$$(2) + 3 \times (1):$$

$$24 = 6B$$

$$B = 4$$

Substitute $B = 4$ in (1): $6 = A + 4$

$$A = 2$$

$$\text{Hence } \frac{6x+6}{x^2-9} \equiv \frac{2}{(x+3)} + \frac{4}{(x-3)}$$

f First factorise the denominator:

$$\frac{7-3x}{x^2-3x-4} \equiv \frac{7-3x}{(x-4)(x+1)}$$

$$\begin{aligned} \text{Then } \frac{7-3x}{(x-4)(x+1)} &\equiv \frac{A}{(x-4)} + \frac{B}{(x+1)} \\ &\equiv \frac{A(x+1) + B(x-4)}{(x-4)(x+1)} \\ 7-3x &\equiv A(x+1) + B(x-4) \end{aligned}$$

Let $x = -1$:

$$7-3 \times (-1) = A(-1+1) + B(-1-4)$$

$$10 = B \times -5$$

$$B = -2$$

1 f (continued)

Let $x = 4$:

$$7 - 3 \times 4 = A(4 + 1) + B(4 - 4)$$

$$-5 = A \times 5$$

$$A = -1$$

$$\text{Hence } \frac{7 - 3x}{x^2 - 3x - 4} \equiv -\frac{1}{(x - 4)} - \frac{2}{(x + 1)}$$

g First factorise the denominator:

$$\frac{8 - x}{x^2 + 4x} \equiv \frac{8 - x}{x(x + 4)}$$

$$\begin{aligned} \text{Then } \frac{8 - x}{x(x + 4)} &\equiv \frac{A}{x} + \frac{B}{(x + 4)} \\ &\equiv \frac{A(x + 4) + Bx}{x(x + 4)} \\ 8 - x &\equiv A(x + 4) + Bx \end{aligned}$$

Let $x = 0$:

$$8 - 0 = A(0 + 4) + B \times 0$$

$$8 = 4A$$

$$A = 2$$

Let $x = -4$:

$$8 - (-4) = A(-4 + 4) + B \times (-4)$$

$$12 = -4B$$

$$B = -3$$

$$\text{Hence } \frac{8 - x}{x^2 + 4x} \equiv \frac{2}{x} - \frac{3}{(x + 4)}$$

1 h First factorise the denominator:

$$\frac{2x-14}{x^2+2x-15} \equiv \frac{2x-14}{(x+5)(x-3)}$$

$$\begin{aligned} \text{Then } \frac{2x-14}{(x+5)(x-3)} &\equiv \frac{A}{x+5} + \frac{B}{x-3} \\ &\equiv \frac{A(x-3) + B(x+5)}{(x+5)(x-3)} \\ 2x-14 &\equiv A(x-3) + B(x+5) \end{aligned}$$

Let $x = 3$:

$$2 \times 3 - 14 = A(3-3) + B(3+5)$$

$$-8 = B \times 8$$

$$B = -1$$

Let $x = -5$:

$$2 \times (-5) - 14 = A(-5-3) + B(-5+5)$$

$$-24 = A \times (-8)$$

$$A = 3$$

$$\text{Hence } \frac{2x-14}{x^2+2x-15} \equiv \frac{3}{x+5} - \frac{1}{x-3}$$

$$\begin{aligned} 2 \quad \frac{-2x-5}{(4+x)(2-x)} &\equiv \frac{A}{4+x} + \frac{B}{2-x} \\ &\equiv \frac{A(2-x) + B(4+x)}{(4+x)(2-x)} \\ -2x-5 &\equiv A(2-x) + B(4+x) \end{aligned}$$

Let $x = 2$:

$$-2 \times 2 - 5 = A(2-2) + B(4+2)$$

$$-9 = B \times 6$$

$$B = -\frac{3}{2}$$

Let $x = -4$:

$$-2 \times (-4) - 5 = A(2 - (-4)) + B(4 + (-4))$$

$$3 = A \times 6$$

$$\frac{1}{2} = A$$

$$\text{Hence } \frac{-2x-5}{(4+x)(2-x)} \equiv \frac{1}{2(4+x)} - \frac{3}{2(2-x)}$$

$$\begin{aligned}
 3 \quad \frac{A}{(x-4)(x+8)} &\equiv \frac{2}{x-4} + \frac{B}{x+8} \\
 &\equiv \frac{2(x+8) + B(x-4)}{(x-4)(x+8)} \\
 A &\equiv 2(x+8) + B(x-4)
 \end{aligned}$$

Let $x = 4$:

$$\begin{aligned}
 A &= 2(4+8) + B(4-4) \\
 &= 24
 \end{aligned}$$

Let $x = -8$:

$$\begin{aligned}
 24 &= 2(-8+8) + B(-8-4) \\
 &= -12B \\
 \Rightarrow B &= -2
 \end{aligned}$$

$$A = 24, B = -2$$

$$\begin{aligned}
 4 \quad \frac{2x^2 - 12x - 26}{(x+1)(x-2)(x+5)} &\equiv \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{x+5} \\
 &\equiv \frac{A(x-2)(x+5) + B(x+1)(x+5) + C(x+1)(x-2)}{(x+1)(x-2)(x+5)} \\
 2x^2 - 12x - 26 &\equiv A(x-2)(x+5) + B(x+1)(x+5) + C(x+1)(x-2)
 \end{aligned}$$

Let $x = -1$:

$$\begin{aligned}
 2 + 12 - 26 &= A \times (-3) \times 4 + 0 + 0 \\
 -12 &= -12A \\
 A &= 1
 \end{aligned}$$

Let $x = 2$:

$$\begin{aligned}
 8 - 24 - 26 &= 0 + B \times 3 \times 7 + 0 \\
 -42 &= 21B \\
 B &= -2
 \end{aligned}$$

Let $x = -5$:

$$\begin{aligned}
 50 + 60 - 26 &= 0 + 0 + C \times (-4) \times (-7) \\
 84 &= 28C \\
 C &= 3
 \end{aligned}$$

$$A = 1, B = -2, C = 3$$

$$\begin{aligned}
 5 \quad \frac{-10x^2 - 8x + 2}{x(2x+1)(3x-2)} &\equiv \frac{D}{x} + \frac{E}{2x+1} + \frac{F}{3x-2} \\
 &\equiv \frac{D(2x+1)(3x-2) + Ex(3x-2) + Fx(2x+1)}{x(2x+1)(3x-2)} \\
 -10x^2 - 8x + 2 &\equiv D(2x+1)(3x-2) + Ex(3x-2) + Fx(2x+1)
 \end{aligned}$$

Let $x = 0$:

$$\begin{aligned}
 2 &= D \times 1 \times (-2) + 0 + 0 \\
 &= -2D \\
 \Rightarrow D &= -1
 \end{aligned}$$

Let $x = -\frac{1}{2}$:

$$\begin{aligned}
 -\frac{5}{2} + 4 + 2 &= 0 + E \times \left(-\frac{1}{2}\right) \times \left(-\frac{7}{2}\right) + 0 \\
 \frac{7}{2} &= \frac{7}{4}E \\
 \Rightarrow E &= 2
 \end{aligned}$$

Let $x = \frac{2}{3}$:

$$\begin{aligned}
 -\frac{40}{9} - \frac{16}{3} + 2 &= 0 + 0 + F \times \left(\frac{2}{3}\right) \times \left(\frac{7}{3}\right) \\
 -\frac{70}{9} &= \frac{14}{9}F \\
 F &= -5
 \end{aligned}$$

$$D = -1, E = 2, F = -5$$

$$\begin{aligned}
 6 \text{ a } \frac{2x^2 - 12x - 26}{(x+1)(x-2)(x+5)} &\equiv \frac{A}{(x+1)} + \frac{B}{(x-2)} + \frac{C}{(x+5)} \\
 &\equiv \frac{A(x-2)(x+5) + B(x+1)(x+5) + C(x+1)(x-2)}{(x+1)(x-2)(x+5)} \\
 2x^2 - 12x - 26 &\equiv A(x-2)(x+5) + B(x+1)(x+5) + C(x+1)(x-2)
 \end{aligned}$$

Let $x = 2$:

$$8 - 24 - 26 = A \times 0 + B \times 3 \times 7 + C \times 0$$

$$-42 = 21B$$

$$B = -2$$

Let $x = -1$:

$$2 + 12 - 26 = A \times (-3) \times 4 + B \times 0 + C \times 0$$

$$-12 = -12A$$

$$A = 1$$

Let $x = -5$:

$$50 + 60 - 26 = A \times 0 + B \times 0 + C \times 28$$

$$84 = 28C$$

$$C = 3$$

$$\text{Hence } \frac{2x^2 - 12x - 26}{(x+1)(x-2)(x+5)} \equiv \frac{1}{(x+1)} - \frac{2}{(x-2)} + \frac{3}{(x+5)}$$

$$\begin{aligned}
 \text{b Let } \frac{-10x^2 - 8x + 2}{x(2x+1)(3x-2)} &\equiv \frac{A}{x} + \frac{B}{(2x+1)} + \frac{C}{(3x-2)} \\
 &\equiv \frac{A(2x+1)(3x-2) + Bx(3x-2) + Cx(2x+1)}{x(2x+1)(3x-2)} \\
 -10x^2 - 8x + 2 &\equiv A(2x+1)(3x-2) + Bx(3x-2) + Cx(2x+1)
 \end{aligned}$$

Let $x = 0$:

$$-0 - 0 + 2 = A \times 1 \times (-2) + B \times 0 + C \times 0$$

$$2 = -2A$$

$$A = -1$$

Let $x = -\frac{1}{2}$:

$$-\frac{10}{4} + 4 + 2 = A \times 0 + B \times \left(-\frac{1}{2}\right) \times \left(-\frac{7}{2}\right) + C \times 0$$

$$\frac{7}{2} = B \times \frac{7}{4}$$

$$\Rightarrow B = 2$$

6 b (continued)

Equate coefficients in x^2 :

$$\begin{aligned} -10 &= 6A + 3B + 2C \\ &= -6 + 6 + 2C \\ &= 2C \\ \Rightarrow C &= -5 \end{aligned}$$

$$\begin{aligned} \text{c Let } \frac{-5x^2 - 19x - 32}{(x+1)(x+2)(x-5)} &\equiv \frac{A}{(x+1)} + \frac{B}{(x+2)} + \frac{C}{(x-5)} \\ &\equiv \frac{A(x+2)(x-5) + B(x+1)(x-5) + C(x+1)(x+2)}{(x+1)(x+2)(x-5)} \\ -5x^2 - 19x - 32 &\equiv A(x+2)(x-5) + B(x+1)(x-5) + C(x+1)(x+2) \end{aligned}$$

Let $x = -1$:

$$\begin{aligned} -5 + 19 - 32 &= A \times 1 \times (-6) + B \times 0 + C \times 0 \\ -18 &= -6A \\ A &= 3 \end{aligned}$$

Let $x = 5$:

$$\begin{aligned} -125 - 95 - 32 &= A \times 0 + B \times 0 + C \times 6 \times 7 \\ -252 &= 42C \\ C &= -6 \end{aligned}$$

Let $x = -2$:

$$\begin{aligned} -20 + 38 - 32 &= A \times 0 + B \times (-1) \times (-7) + C \times 0 \\ -14 &= 7B \\ B &= -2 \end{aligned}$$

$$\text{Hence } \frac{-5x^2 - 19x - 32}{(x+1)(x+2)(x-5)} \equiv \frac{3}{(x+1)} - \frac{2}{(x+2)} - \frac{6}{(x-5)}$$

7 a First factorise the denominator:

$$\frac{6x^2 + 7x - 3}{x^3 - x} \equiv \frac{6x^2 + 7x - 3}{x(x+1)(x-1)}$$

$$\begin{aligned} \text{Then } \frac{6x^2 + 7x - 3}{x(x+1)(x-1)} &\equiv \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1} \\ &\equiv \frac{A(x+1)(x-1) + Bx(x-1) + Cx(x+1)}{x(x+1)(x-1)} \\ 6x^2 + 7x - 3 &\equiv A(x+1)(x-1) + Bx(x-1) + Cx(x+1) \end{aligned}$$

7 a (continued)

Let $x = 0$:

$$\begin{aligned} -3 &= A \times 1 \times (-1) + 0 + 0 \\ &= -A \\ \Rightarrow A &= 3 \end{aligned}$$

Let $x = -1$:

$$\begin{aligned} 6 - 7 - 3 &= 0 + B \times (-1) \times (-2) + 0 \\ -4 &= 2B \\ \Rightarrow B &= -2 \end{aligned}$$

Let $x = 1$:

$$\begin{aligned} 6 + 7 - 3 &= 0 + 0 + C \times 1 \times 2 \\ 10 &= 2C \\ \Rightarrow C &= 5 \end{aligned}$$

$$\text{So } \frac{6x^2 + 7x - 3}{x(x+1)(x-1)} \equiv \frac{3}{x} - \frac{2}{x+1} + \frac{5}{x-1}$$

b First factorise the denominator:

$$\frac{8x+9}{10x^2+3x-4} \equiv \frac{8x+9}{(5x+4)(2x-1)}$$

$$\begin{aligned} \text{Then } \frac{8x+9}{(5x+4)(2x-1)} &\equiv \frac{A}{5x+4} + \frac{B}{2x-1} \\ &\equiv \frac{A(2x-1)}{(5x+4)(2x-1)} + \frac{B(5x+4)}{(5x+4)(2x-1)} \\ 8x+9 &\equiv A(2x-1) + B(5x+4) \end{aligned}$$

Let $x = -\frac{4}{5}$:

$$\begin{aligned} -\frac{32}{5} + 9 &= A \times \left(-\frac{13}{5}\right) + 0 \\ \frac{13}{5} &= -\frac{13}{5}A \\ \Rightarrow A &= -1 \end{aligned}$$

Let $x = \frac{1}{2}$:

$$\begin{aligned} 4 + 9 &= 0 + B \times \left(\frac{13}{2}\right) \\ 13 &= \frac{13}{2}B \\ \Rightarrow B &= 2 \end{aligned}$$

$$\text{So } \frac{8x+9}{(5x+4)(2x-1)} \equiv -\frac{1}{5x+4} + \frac{2}{2x-1}$$

Challenge

Evaluating the denominator at $x = 2$:

$$2^3 - 4(2^2) + 2 + 6 = 0$$

By the factor theorem, $(x-2)$ is a factor of $x^3 - 4x^2 + x + 6$

So we can write $x^3 - 4x^2 + x + 6 = (x-2) \times p(x)$ for some quadratic polynomial p .

We can find p using long division:

$$\begin{array}{r}
 x^2 - 2x - 3 \\
 x-2 \overline{) x^3 - 4x^2 + x + 6} \\
 \underline{x^3 - 2x^2} \\
 -2x^2 + x \\
 \underline{-2x^2 + 4x} \\
 -3x + 6 \\
 \underline{-3x + 6} \\
 0
 \end{array}$$

$$\begin{aligned}
 \frac{5x^2 - 15x - 8}{x^3 - 4x^2 + x + 6} &\equiv \frac{5x^2 - 15x - 8}{(x-2)(x^2 - 2x - 3)} \\
 &\equiv \frac{5x^2 - 15x - 8}{(x-2)(x+1)(x-3)} \\
 &\equiv \frac{A}{x-2} + \frac{B}{x+1} + \frac{C}{x-3} \\
 &\equiv \frac{A(x+1)(x-3) + B(x-2)(x-3) + C(x-2)(x+1)}{(x-2)(x+1)(x-3)}
 \end{aligned}$$

$$5x^2 - 15x - 8 \equiv A(x+1)(x-3) + B(x-2)(x-3) + C(x-2)(x+1)$$

Let $x = 2$:

$$20 - 30 - 8 = A \times 3 \times (-1) + 0 + 0$$

$$-18 = -3A$$

$$\Rightarrow A = 6$$

Let $x = -1$:

$$5 + 15 - 8 = 0 + B \times (-3) \times (-4) + 0$$

$$12 = 12B$$

$$\Rightarrow B = 1$$

Let $x = 3$:

$$45 - 45 - 8 = 0 + 0 + C \times 1 \times 4$$

$$-8 = 4C$$

$$C = -2$$

$$\text{So } \frac{5x^2 - 15x - 8}{x^3 - 4x^2 + x + 6} \equiv \frac{6}{x-2} + \frac{1}{x+1} - \frac{2}{x-3}$$