Core 4 June 2009 Solutions

1)
$$
\begin{array}{r}
3x^2 \quad -4x-5 \\
x^2+x+2 \overline{)\,3x^4 \quad -x^3-3x^2-14x-8} \\
\underline{3x^4+3x^3+6x^2} \\
-4x^3-9x^2-14x \\
\underline{-4x^3-4x^2-\ 8x} \\
-5x^2-6x-8 \\
\underline{-5x^2-5x-10} \\
-x\ +2
\end{array}
$$

Quotient = $3x^2$ -4x -5; Remainder= -x+2 [4]

2) Let x= $\tan\theta$ $\therefore$ dx = $\sec^2\theta$ dθ when x = 1, $\theta = \tan^{-1}1 = \frac{\pi}{4}$; when x = $\sqrt{3}$, $\theta = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$

$$\int_1^{\sqrt{3}} \frac{1-x^2}{1+x^2}dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1-\tan^2\theta}{1+\tan^2\theta}\sec^2\theta d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (1-\tan^2\theta)d\theta \quad (\because \sec^2\theta \equiv 1+\tan^2\theta)$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (1-\{\sec^2\theta-1\})d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (2-\sec^2\theta)d\theta = [2\theta-\tan\theta]_{\frac{\pi}{4}}^{\frac{\pi}{3}} = (\frac{2\pi}{3}-\tan\frac{\pi}{3})-(\frac{2\pi}{4}-\tan\frac{\pi}{4})$$

$$= (\frac{2\pi}{3}-\sqrt{3})-(\frac{\pi}{2}-1) = \quad \frac{2\pi}{3}-\frac{\pi}{2}-\sqrt{3}+1 \quad = \frac{\pi}{6}-\sqrt{3}+1$$ [7]

3)(i)

$$(a+x)^{-2} = a^{-2}\left(1+\frac{x}{a}\right)^{-2} = a^{-2}\{1+\ -2\ \frac{x}{a}+\frac{-2\times -3}{2!}\left(\frac{x}{a}\right)^2 + ...\} = \frac{1}{a^2}-\frac{2x}{a^3}+\frac{3x^2}{a^4}$$ *neglecting higher powers.*[3]

(ii) $(1-x)(a+x)^{-2} \approx (1-x)\left(\frac{1}{a^2}-\frac{2x}{a^3}+\frac{3x^2}{a^4}\right)$

The term in x^2 is given by $1\times\frac{3x^2}{a^4} - x\times\left(-\frac{2x}{a^3}\right) = \frac{3x^2}{a^4}+\frac{2x^2}{a^3} = \left(\frac{3}{a^4}+\frac{2}{a^3}\right)x^2$

The coefficient of this term is 0, $\therefore \left(\frac{3}{a^4}+\frac{2}{a^3}\right) = 0$ Multiplying by a^4 *gives* $3+2a=0$. *Hence* $a = -\frac{3}{2}$.[4]

4)(i) Let y= e^x(sin2x-2cos2x)

Then using the Product Rule $\frac{d(UV)}{dx} = U\frac{dV}{dx}+V\frac{dU}{dx}$ and the Chain Rule $\frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx}$

$$\frac{dy}{dx} = e^x(2\cos 2x - 2\times 2\sin 2x) + (\sin 2x - 2\cos 2x)e^x \quad \{\because \frac{d}{dx}(\sin x) = \cos x, \frac{d}{dx}(\cos x) = -\sin x \ \&\ \frac{d}{dx}(e^x) = e^x\}$$

$$\frac{dy}{dx} = e^x\{2\cos 2x + 4\sin 2x + \sin 2x - 2\cos 2x) = e^x\{5\sin 2x\} = 5e^x\sin 2x$$ [4]

(ii) $\int_0^{\frac{\pi}{4}} e^x\sin 2x dx = \frac{1}{5}\int_0^{\frac{\pi}{4}} 5e^x\sin 2x dx = \frac{1}{5}[e^x(\sin 2x - 2\cos 2x)]_0^{\frac{\pi}{4}}$. by reversing part(i)

$$= \frac{1}{5}\{e^{\frac{\pi}{4}}(\sin\frac{\pi}{2}-2\cos\frac{\pi}{2}) - e^0(\sin 0 - 2\cos 0)\} = \frac{1}{5}\{e^{\frac{\pi}{4}}\times 1 - 1\times -2) \qquad (\because \sin\frac{\pi}{2}=1, \cos\frac{\pi}{2}=0, e^0=1, \sin 0 = 0, \cos 0 = 1).$$

$$= \frac{1}{5}\left(e^{\frac{\pi}{4}}+2\right)$$ [3]

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5)(i) x = 2t + t². $\therefore \frac{dx}{dt} = 2 + 2t$. y = 2t² + t³. $\therefore \frac{dy}{dt} = 4t + 3t^2$.

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 4t + 3t^2 \times \frac{1}{2+2t} = \frac{4t+3t^2}{2+2t}$ (Chain Rule & $\frac{dt}{dx} = 1 \Big/ \frac{dx}{dt}$)

y = 2t² + t³ = (2t + t²)t = xt $\therefore$ at (3,–9) –9 = 3t. $\therefore$ t = –3 at (3,–9).

$\therefore$ gradient at (3,–9) is $\frac{4(-3)+3(-3)^2}{2+2(-3)} = \frac{-12+27}{2-6} = \frac{15}{-4} = -3.75$. [5]

(ii) $\frac{y}{x} = \frac{2t^2+t^3}{2t+t^2} = \frac{(2t+t^2)t}{2t+t^2} = t$. Substituting into the formula for x gives $x = \frac{2y}{x} + \frac{y^2}{x^2}$

Multiplying by x² gives x³ = 2xy + y², which rearranges to x³–2xy–y²=0 as the Cartesian equation. 4]

6)(i) Let $f(x) \equiv \frac{4x}{(x-5)(x-3)^2} \equiv \frac{A}{x-5} + \frac{B}{(x-3)} + \frac{C}{(x-3)^2}$

Multiplying by (x–5)(x–3)² gives $4x \equiv A(x-3)^2 + B(x-3)(x-5) + C(x-5)$

Putting x=5 gives 20 = 4A +0+0. $\therefore$ A = 5

Putting x=3 gives 12 = 0+0 –2C $\therefore$ C = –6

Putting x=0 gives 0 = 9A +15B -5C = 45 +15B +30 = 75 + 15B $\therefore$ B= –5

$\therefore f(x) \equiv \frac{5}{x-5} - \frac{5}{(x-3)} - \frac{6}{(x-3)^2}$ [4]

(ii) $\int_1^2 f(x)dx = \int_1^2 (\frac{5}{x-5} - \frac{5}{(x-3)} - \frac{6}{(x-3)^2})dx = [5\ln|x-5| - 5\ln|x-3| + 6(x-3)^{-1}]_1^2$

$= (5\ln|-3| - 5\ln|-1| + 6(-1)^{-1}) - (5\ln|-4| - 5\ln|-2| + 6(-2)^{-1}) = (5\ln 3 - 0 - 6) - (5\ln 4 - 5\ln 2 - 3)$

= 5ln3 – 6 – 5ln2 + 3 = $5\ln\frac{3}{2} - 3 = 5\ln\frac{3}{2} - 3$. [5]

7) (i) If **u** is perpendicular to 4i + k and to 4i +3j + 2k, then $\left(\frac{3}{13}\underline{i} + b\underline{j} + c\underline{k}\right) \bullet (4\underline{i} + \underline{k}) = 0$

& $\left(\frac{3}{13}\underline{i} + b\underline{j} + c\underline{k}\right) \bullet (4\underline{i} + 3\underline{j} + 2\underline{k}) = 0$

The first gives $\frac{12}{13} + c = 0$, *hence* $c = -\frac{12}{13}$. The second gives $\frac{12}{13} + 3b + 2c = 0$,

hence $\frac{12}{13} + 3b - \frac{24}{13} = 0$, $3b = \frac{12}{13}$, $\therefore b = \frac{4}{13}$. *So* $\underline{u} = \frac{3}{13}\underline{i} + \frac{4}{13}\underline{j} - \frac{12}{13}\underline{k}$

$|\underline{u}| = \sqrt{\left(\frac{3}{13}\right)^2 + \left(\frac{4}{13}\right)^2 + \left(-\frac{12}{13}\right)^2} = \sqrt{\frac{9+16+144}{169}} = 1$ $\therefore \underline{u}$ *is a unit vector.* [6]

(ii) $(4\underline{i} + \underline{k}) \bullet (4\underline{i} + 3\underline{j} + 2\underline{k}) = 4\times4 + 0\times3 + 1\times2 = 18$; $|4\underline{i} + \underline{k}| = \sqrt{4^2 + 0^2 + 1^2} = \sqrt{17}$;

$|4\underline{i} + 3\underline{j} + 2\underline{k}| = \sqrt{4^2 + 3^2 + 2^2} = \sqrt{29}$ $\underline{a} \bullet \underline{b} = |\underline{a}||\underline{b}|\cos\vartheta$ $\therefore \cos\vartheta = \frac{\underline{a} \bullet \underline{b}}{|\underline{a}||\underline{b}|} = \frac{18}{\sqrt{17}\sqrt{29}}$

$\therefore \vartheta = 36^o$ to the nearest degree. [3]

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8)(i) $14x^2 - 7xy + y^2 = 2$ Differentiating implicitly and using the Product Rule (on $-7xy$)gives

$$28x - 7x\frac{dy}{dx} - 7y + 2y\frac{dy}{dx} = 0$$ Rearranging and factorising gives

$$28x - 7y + (2y - 7x)\frac{dy}{dx} = 0 \quad \therefore (2y - 7x)\frac{dy}{dx} = (7y - 28x) \quad \text{Hence } \frac{dy}{dx} = \frac{(7y - 28x)}{(2y - 7x)}$$

Multiplying the Numerator and Denominator of the r.h.s. by –1 gives $\frac{dy}{dx} = \frac{(28x - 7y)}{(7x - 2y)}$

{In questions like this where you have to show a given result it is essential to show all of the working, otherwise you will certainly lose marks.} [4]

(ii) Substituting x = 1 into $14x^2 - 7xy + y^2 = 2$ gives $14 - 7y + y^2 = 2$ Hence $y^2 - 7y + 12 = 0$.

Factorising: (y – 3)(y – 4) = 0 So y = 3 or y = 4. So L is (1,3) and M is (1,4) or vice versa.

$\frac{dy}{dx} = \frac{(28x - 7y)}{(7x - 2y)}$ so at (1,3) the gradient $\frac{dy}{dx} = \frac{(28 - 21)}{(7 - 6)} = 7$

Using $y - y_1 = m(x - x_1)$ the equation of the tangent at (1, 3) is y – 3 = 7(x – 1). So y = 7x – 4

At (1, 4) the gradient $\frac{dy}{dx} = \frac{(28 - 28)}{(7 - 8)} = 0$ So the tangent is the horizontal line y = 4.

At the intersection point N, 4 = 7x – 4. So $x = \frac{8}{7}$. So the co-ordinates of N are $(\frac{8}{7}, 4)$. [6]

(i) $\frac{d\vartheta}{dt} = k_1 \Rightarrow \int d\vartheta = \int k_1 dt \Rightarrow \theta = k_1t + c$ If t = 0 θ = 40 ⇒ c = 40.

If θ = 60 60 = k_1t + 40 ⇒ $t = \frac{20}{k_1}$ So time for increase $= \frac{20}{k_1}$ [1]

(ii) Differential equation is $\frac{d\vartheta}{dt} = -k_2(\vartheta - 20)$ (-ve because it is decreasing.) [1]

(iii) $\frac{d\vartheta}{dt} = -k_2(\vartheta - 20)$ Separating the variables gives $\int \frac{d\vartheta}{(\vartheta - 20)} = \int -k_2 dt$

Integrating each side gives $\ln|\vartheta - 20| = -k_2t + A$ where A is an arbitrary constant.

However at the start of cooling the temperature was 60°C. $\therefore \vartheta = 60 \text{ when } t = 0$.

$\therefore \ln|60 - 20| = -k_2 \times 0 + A$ ∴ A = ln40 Hence $\ln|\vartheta - 20| = -k_2t + \ln 40$

When $\vartheta = 40^\circ C$, $\ln|40 - 20| = -k_2t + \ln 40$ So $k_2t = \ln 40 - \ln 20 = \ln 2$

$\therefore t = \frac{\ln 2}{k_2}$ So the total time for the temperature to rise and fall $= \frac{20}{k_1} + \frac{\ln 2}{k_2}$. [8]