June 2007.

Core 4.

1) $f(x) = \dfrac{3x+1}{(x+2)(x-3)} = \dfrac{A}{(x+2)} + \dfrac{B}{(x-3)}$ $\quad A(x-3) + B(x+2) \equiv 3x+1 \quad 5B = 10 \quad B = 2.$

$-5A = -5 \quad A = 1$

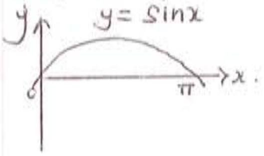
(i) $f(x) = \dfrac{1}{x+2} + \dfrac{2}{x-3}.$ $\quad f'(x) = -\dfrac{1}{(x+2)^2} - \dfrac{2}{(x-3)^2} = -1\underbrace{\left(\dfrac{1}{(x+2)^2} + \dfrac{2}{(x-3)^2}\right)}_{\text{always +ve}}$ hence $f'(x)$ is always negative for all x.

2) $\displaystyle\int_0^1 \overset{u}{x^2}\overset{\frac{dv}{dx}}{e^x}\,dx \quad u = x^2 \quad \frac{du}{dx} = 2x \quad \frac{dv}{dx} = e^x \quad v = e^x \quad \int_0^1 x^2e^x\,dx = x^2e^x - \int e^x 2x\,dx$

$\displaystyle\int \overset{\frac{dv}{dx}}{e^x}\overset{u}{2x}\,dx \quad u = 2x \quad \frac{du}{dx} = 2 \quad \frac{dv}{dx} = e^x \quad v = e^x \quad \int e^x 2x\,dx = 2xe^x - \int e^x 2\,dx$

$\displaystyle\int_0^1 x^2e^x\,dx = \left[x^2e^x - \left(2xe^x - 2e^x\right)\right]_0^1 = \left[x^2e^x - 2xe^x + 2e^x\right]_0^1 = \left(e^1 - 2e^1 + 2e^1\right) - (+2)$

$\displaystyle\int_0^1 x^2e^x\,dx = e^1 - 2.$

3). 

$\cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$

$\sin^2 A = \frac{1}{2}(1 - \cos 2A)$

$\displaystyle V = \pi\int y^2\,dx = \pi\int \sin^2 x\,dx.$

$\displaystyle V = \pi\int \frac{1}{2}(1 - \cos 2x)\,dx. \quad \frac{\pi}{2}\left[x - \frac{\sin 2x}{2}\right]_0^\pi = \frac{\pi}{2}\left[\left(\pi - \frac{1}{2}\sin 2\pi\right) - \left(0 - \frac{\sin 0}{2}\right)\right]$

$= \dfrac{\pi^2}{2}$

4)(i) $(2+x)^{-2} = \left(2\left(1 + \frac{x}{2}\right)\right)^{-2} = \frac{1}{4}\left(1 + \frac{x}{2}\right)^{-2} = \frac{1}{4}\left\{1 + (-2)\left(\frac{x}{2}\right) + \frac{(-2)(-3)}{2!}\left(\frac{x}{2}\right)^2 + \frac{(-2)(-3)(-4)}{3!}\left(\frac{x}{2}\right)^3 + \dots\right\}$

$= \frac{1}{4}\left\{1 - x + \frac{3x^2}{4} - \cancel{4}\frac{x^3}{\cancel{8}_2} + \dots\right\} = \frac{1}{4} - \frac{x}{4} + \frac{3x^2}{16} - \frac{x^3}{8} + \dots$

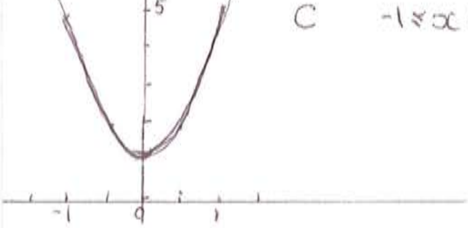
$\left|\frac{x}{2}\right| < 1 \quad -2 < x < 2$

(ii) $(1+x^2)\left\{\frac{1}{4} - \frac{x}{4} + \frac{3x^2}{16} - \frac{x^3}{8} + \dots\right.$ $\quad$ coefficient x^3 $\left(-\frac{1}{4} - \frac{1}{8}\right) = -\frac{3}{8}$

5. $x = \cos t \quad y = 3 + 2\cos 2t. \quad \dfrac{dy}{dx} = \dfrac{4\sin 2t}{-\sin t} = \dfrac{4 \times 2\sin t\cos t}{-\sin t} = +8\cos t.$

$\dfrac{dx}{dt} = -\sin t \quad \dfrac{dy}{dt} = -2\sin 2t \times 2.$ $\quad$ max value of $+8\cos t = 8$ when $\cos t = +1$ $t = 0$

$3 + 2\cos 2t = 3 + 2(2\cos^2 t - 1) = 3 + 4\cos^2 t - 2 = 1 + 4\cos^2 t \quad \underline{y = 1 + 4x^2}$



C $\quad -1 \le x \le 1 \quad 0 \le y \le 5$

6. $x^2 + 3xy + 4y^2 = 58 \quad 2x + 3x\dfrac{dy}{dx} + 3y + 4 \times 2y\dfrac{dy}{dx} = 0$

$(3x + 8y)\dfrac{dy}{dx} + 2x + 3y = 0. \quad \dfrac{dy}{dx} = \dfrac{-(2x+3y)}{3x + 8y}.$ $\quad$ grad tgt $(2,3) = \dfrac{-13}{30}$

grad of normal $+\frac{30}{13}$ at $(2,3)$ $\quad (y-3) = +\frac{30}{13}(x-2)$

$13y - 39 = +30x - 60$ $\quad \underline{30x - 13y - 21 = 0}$.

7). $$\begin{array}{r} 2x + 3 \\ x^2+4 \overline{)\, 2x^3 + 3x^2 + 9x + 12} \\ \underline{2x^3 + 0 + 8x} \\ 3x^2 + x + 12 \\ \underline{3x^2 + 0 + 12} \\ 0 + x + 0 \end{array}$$

quotient $2x+3$ $\quad$ remainder $\frac{x}{x^2+4}$

$$\frac{2x^3+3x^2+9x+12}{x^2+4} = 2x+3 + \frac{x}{x^2+4}$$ $\quad A=2 \quad B=3 \quad C=1 \quad D=0$.

$$\int_1^3 2x+3 + \frac{x}{x^2+4}\,dx = \frac{2x^2}{2} + 3x + \frac{1}{2}\int_1^3 \frac{2x}{x^2+4}\,dx = \left[x^2+3x+\frac{1}{2}\ln|x^2+4|\right]_1^3$$

$$= \left(9+9+\tfrac{1}{2}\ln 13\right) - \left(1+3+\tfrac{1}{2}\ln 5\right) = 14 + \tfrac{1}{2}(\ln 13 - \ln 5) = 14 + \tfrac{1}{2}\ln\tfrac{13}{5}$$

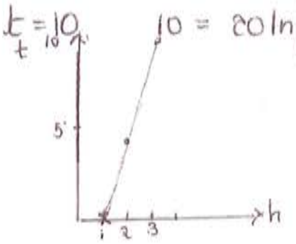
8) $\frac{dh}{dt} = \frac{6-h}{20}$ $\quad \int \frac{1}{6-h}\,dh = \int \frac{1}{20}\,dt$. $\quad \frac{\ln|6-h|}{-1} = \frac{t}{20} + k$.

$\ln|6-h| = -\frac{t}{20} - k$ $\quad t=0 \; h=1$. $-\ln 5 = +k$ $\quad -\ln|6-h| + \ln 5 = +\frac{t}{20}$.

height at planting is 1m

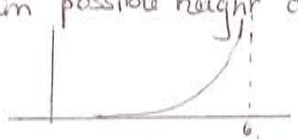
$\underline{20\ln\left|\frac{5}{6-h}\right| = t.}$ $\quad h = 2\text{m} \quad t = 20\ln\frac{5}{4}$ $\quad \underline{t = 4.46 \text{ years (3sf)}}$ $\quad 4.4629$

$t = 10$ $\quad 10 = 20\ln\left|\frac{5}{6-h}\right|$ $\quad e^{\frac{1}{2}} = \frac{5}{6-h}$ $\quad 6-h = \frac{5}{\sqrt{e}} = 3.03265$ $\quad h = 2.9673$



$\underline{h = 2.97\text{m (3sf)}}$

maximum possible height of shrub is 6m.



9(i) $l_1 \cdot l_2 = \begin{pmatrix}-6\\8\\-2\end{pmatrix} \cdot \begin{pmatrix}1\\3\\2\end{pmatrix} = -6+24-4 = 14$. $\quad |l_1||l_2|\cos\theta = 14$.

$$\cos\theta = \frac{14}{\sqrt{36+64+4} \times \sqrt{1+9+4}} = \frac{14}{\sqrt{104}\sqrt{14}} = \frac{14}{\sqrt{16\times 91}} = \frac{14}{4\sqrt{91}}$$

(ii) $\begin{pmatrix}-6\\8\\-2\end{pmatrix} = -2\begin{pmatrix}3\\c\\1\end{pmatrix}$ $\quad \therefore \underline{c = -4}$

parallel.

(iii) L_2 L_3 intersect $\quad \begin{pmatrix}3\\-8\\0\end{pmatrix} + t\begin{pmatrix}1\\3\\2\end{pmatrix} = \begin{pmatrix}2\\1\\3\end{pmatrix} + u\begin{pmatrix}3\\c\\1\end{pmatrix}$

(x) $3+t = 2+3u$ $\quad t = 3u-1 \quad 2t = 6u-2 = 3+u \quad 5u = 5 \; u=1 \quad t=2$

(z) $2t = 3+u$

(y) $-8+6 = 1+c$ $\quad \underline{c = -3}$