

June 2006 C4

1) $80x + 2x \frac{dy}{dx} + 2y + 2y \frac{dy}{dx} = 0$ $\frac{dy}{dx} = \frac{-8x-2y}{2x+2y} = \frac{-(4x+y)}{x+y}$
 at (1,2) $\frac{dy}{dx} = \frac{-6}{3} = -2$

2) $(1-3x)^{-2} = 1 + -2(-3x) + \frac{(-2)(-3)(-3x)^2}{2} + \dots = 1 + 6x + 27x^2 + \dots$

ii) $(1+2x)^2 = 1 + 4x + 4x^2$

$(1+2x)^2(1-3x)^{-2} = (1+4x+4x^2)(1+6x+27x^2 + \dots)$

coefficient x^2 $27x^2 + 24x^2 + 4x^2$ $55x^2$ coefficient x^2 is 55

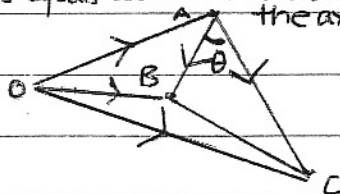
3) $\frac{3-2x}{x(3-x)} \equiv \frac{A}{x} + \frac{B}{3-x}$ $3-2x \equiv A(3-x) + Bx$ $x=0$ $3=3A$ $A=1$
 $x=3$ $-3=3B$ $B=-1$

$\frac{3-2x}{x(3-x)} = \frac{1}{x} - \frac{1}{3-x}$

$\int_1^2 \frac{3-2x}{x(3-x)} dx = \int_1^2 \left(\frac{1}{x} - \frac{1}{3-x} \right) dx = \left[\ln|x| + \ln|3-x| \right]_1^2 = (\ln 2 + \ln 1) - (\ln 1 + \ln 2) = 0$

the area $x=1$ to $x=1.5$ above x -axis equals area $x=1.5$ to $x=2$ below the axis.

4 $\vec{AB} = \underline{b} - \underline{a} = \begin{pmatrix} 4 \\ 2 \\ -4 \end{pmatrix} - \begin{pmatrix} 7 \\ 3 \\ -3 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \\ -1 \end{pmatrix}$ $\vec{AC} = \underline{c} - \underline{a} = \begin{pmatrix} 5 \\ 4 \\ -5 \end{pmatrix} - \begin{pmatrix} 7 \\ 3 \\ -3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix}$



(i) $AB \cdot AC = |AB| |AC| \cos \theta$ $\cos \theta = \frac{\begin{pmatrix} -3 \\ -1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix}}{\sqrt{9+1+1} \sqrt{4+1+4}}$

$\cos \theta = \frac{6 + -1 + 2}{\sqrt{11} \sqrt{9}} = \frac{7}{3\sqrt{11}}$ $\theta = 45.3^\circ$ (3sf)

(ii) $\cos \theta = \frac{7}{3\sqrt{11}}$ $\cos^2 \theta = \frac{49}{9 \times 11}$ $\sin^2 \theta = 1 - \cos^2 \theta = 1 - \frac{49}{99} = \frac{50}{99}$ $\sin \theta = \frac{5\sqrt{2}}{3\sqrt{11}}$

Area = $\frac{1}{2} \sqrt{11} \times 3 \sin \theta = \frac{3\sqrt{11} \times 5\sqrt{2}}{2 \times 3\sqrt{11}} = \frac{5\sqrt{2}}{2}$

5. $\frac{dA}{dt} = kA^2$ $\int \frac{1}{A^2} dA = \int k dt$ $-\frac{1}{A} = kt + C$ $-\frac{1}{1000} = k + C$ $-\frac{1}{2000} = 2k + C$

$-\frac{1}{2000} - \left(-\frac{1}{1000}\right) = k$ $k = \frac{1}{2000}$ $-\frac{1}{1000} = \frac{1}{2000} + C$ $C = -\frac{3}{2000}$

$-\frac{1}{A} = \frac{1}{2000} t - \frac{3}{2000}$ $A = 3000$ $-\frac{1}{3000} = \frac{1}{2000} (t-3)$ $-\frac{2}{3} = t-3$

$t = 2\frac{1}{3}$ hours

6. $u = e^x + 1$ $\frac{du}{dx} = e^x$ $dx = e^{-x}$ $e^x = u-1$ $\frac{dx}{du} = \frac{1}{u-1}$
 $\int \frac{e^{2x}}{e^x+1} dx = \int \frac{(e^x)^2}{e^x+1} dx = \int \frac{(u-1)^2}{u} \times \frac{1}{(u-1)} du = \int \frac{(u-1)}{u} du$

$x=1$ $u = e^1 + 1$ $x=0$ $u = 2$

$$\int_0^1 \frac{e^{2x}}{e^{x+1}} dx = \int_2^{e^1+1} \frac{u-1}{u} du = \int_2^{e^1+1} \left(1 - \frac{1}{u}\right) du = \left[u - \ln|u| \right]_2^{e^1+1}$$

$$(e^1+1 - \ln|e^1+1|) - (2 - \ln|2|) = e^1 - 1 - (\ln|e^1+1| - \ln|2|) = e^1 - 1 - \ln\left(\frac{e^1+1}{2}\right)$$

7. $r = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ a \end{pmatrix}$ $r = \begin{pmatrix} -8 \\ 2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$ lines are not parallel.

Skew lines do not intersect. $1 + 3\lambda = -8 + \mu$ (x)

$$-2 + \lambda = 2 - 2\mu$$
 (y)

$$4 + a\lambda = 3 - \mu$$
 (z)

Use x and y to find λ and μ . $9 = \mu - 3\lambda$ $18 = 2\mu - 6\lambda$ $14 = -7\lambda$ $\lambda = -2$
 $-4 = -2\mu - \lambda$ $-4 = -2\mu - 2$ $\mu = 3$

(i) for lines to intersect $4 + -2a = 3 - 3$ $a = 2$ lines are skew when $a \neq 2$.

(ii) $a = 2$. $\begin{pmatrix} 1-6 \\ -2-2 \\ 4-4 \end{pmatrix} = \begin{pmatrix} -5 \\ -4 \\ 0 \end{pmatrix}$ check. $\begin{pmatrix} -8+3 \\ 2-6 \\ 3-3 \end{pmatrix} = \begin{pmatrix} -5 \\ -4 \\ 0 \end{pmatrix}$ is point of intersection.

8. $\cos^2 x = 2\cos^2 x - 1$ $\cos^2 x = \frac{1}{2}(\cos 2x + 1)$ $\int \cos^2 6x dx = \int \frac{1}{2}(\cos 12x + 1) dx$.

$$\frac{1}{2} \int \cos 12x + 1 dx = \frac{1}{2} \left(\frac{\sin 12x}{12} + x \right) + c = \frac{1}{2}x + \frac{\sin 12x}{24} + c$$

$$\int_0^{\frac{\pi}{12}} x \cos^2 6x dx = x \left(\frac{x}{2} + \frac{\sin 12x}{24} \right) - \int \frac{x}{2} + \frac{\sin 12x}{24} dx$$

$u = x$ $\frac{du}{dx} = 1$
 $\frac{dv}{dx} = \cos^2 6x$ $v = \frac{x}{2} + \frac{\sin 12x}{24}$

$$= \left[x \left(\frac{x}{2} + \frac{\sin 12x}{24} \right) - \left(\frac{x^2}{4} + \frac{-\cos 12x}{12 \times 24} \right) \right]_0^{\frac{\pi}{12}} = \left(\frac{\pi}{12} \left(\frac{\pi}{24} + 0 \right) - \left(\frac{\pi^2}{4 \times 144} + \frac{1}{12 \times 24} \right) \right) - (0 - \frac{0-1}{12 \times 24})$$

$$= \frac{\pi^2}{12 \times 24} - \frac{\pi^2}{4 \times 144} - \frac{1}{12 \times 24} - \frac{1}{12 \times 24} = \frac{1}{576} (\pi^2 - 4)$$

9. $x = 4 \cos t$ $y = 3 \sin t$ $\frac{dx}{dt} = -4 \sin t$ $\frac{dy}{dt} = 3 \cos t$ $\frac{dy}{dx} = \frac{3 \cos t}{-4 \sin t}$
 $\frac{dy}{dx} = -\frac{3}{4} \cot t$ $t = p$ $y - 3 \sin p = \frac{-3 \cos p}{4 \sin p} (x - 4 \cos p)$

$$4y \sin p - 12 \sin^2 p = -3x \cos p + 12 \cos^2 p$$
 $4y \sin p + 3x \cos p = 12$

at R $y=0$ $x = \frac{4}{\cos p}$ at S $x=0$ $y = \frac{3}{\sin p}$ area of triangle $\frac{1}{2}xy$

$$\text{area} = \frac{1}{2} \frac{4}{\cos p} \frac{3}{\sin p} = \frac{12}{\sin 2p}$$

least possible value of area of triangle is 12 when $\sin 2p = 1$ $2p = \frac{\pi}{2}$

when $p = \frac{\pi}{4}$