# C4 Solutions
# June 2005

1

$$
\begin{array}{r}
x^2 + 2x + 2 \qquad\qquad \\
x^2 + x + 1\overline{\big)\, x^4 + 3x^3 + 5x^2 + 4x - 1} \\
\underline{x^4 + x^3 + x^2} \quad \downarrow \qquad \\
2x^3 + 4x^2 + 4x \quad \\
\underline{2x^3 + 2x^2 + 2x} \quad \\
2x^2 \quad 2x \quad 1 \quad - \\
\underline{2x^2 \quad 2x \quad 2} \quad + \\
-3 \quad\;\;
\end{array}
$$

The quotient is $x^2+2x+2$ and the remainder is -3.

2 We use integration by parts with

$$u = x \qquad \Rightarrow \qquad \frac{du}{dx} = 1$$

$$\frac{dv}{dx} = \cos x \qquad \Rightarrow \qquad v = \sin x$$

Therefore

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx$$

$$= x\sin x - (-\cos x)$$

$$= x\sin x + \cos x$$

So, $\displaystyle\int_0^{\frac{\pi}{2}} x\cos x\,dx = \Big[x\sin x + \cos x\Big]_0^{\frac{\pi}{2}} = (\tfrac{\pi}{2}+0)-(0+1) = \tfrac{\pi}{2}-1$

3 (i) The direction of line $L_1$ is $\begin{pmatrix}-1\\-2\\-4\end{pmatrix} - \begin{pmatrix}2\\-3\\1\end{pmatrix} = \begin{pmatrix}-3\\1\\-5\end{pmatrix}$ (or you could subtract the vectors the other way round!)

The equation for $L_1$ can be found using a point on the line and the direction. So a suitable equation would be:

$$\mathbf{r} = \begin{pmatrix}2\\-3\\1\end{pmatrix} + t\begin{pmatrix}-3\\1\\-5\end{pmatrix}$$

(or you could use the point (-1, -2, -4) as the point the line passes through).

(ii) The equation for $L_2$ is

$$\mathbf{r} = \begin{pmatrix}3\\2\\-9\end{pmatrix} + s\begin{pmatrix}4\\-4\\5\end{pmatrix}$$

(note the need for different letters in the equations for the two lines.

We need to show that the lines do not meet.
If the lines were to meet the point of intersection is located where:

$$\begin{pmatrix}2\\-3\\1\end{pmatrix}+t\begin{pmatrix}-3\\1\\-5\end{pmatrix}=\begin{pmatrix}3\\2\\-9\end{pmatrix}+s\begin{pmatrix}4\\-4\\5\end{pmatrix}$$

We have 3 equations:

| | | | |
|---|---|---|---|
| 2 – 3t = 3 + 4s | so | 3t + 4s = -1 | (1) |
| -3 + t = 2 – 4s | so | t + 4s = 5 | (2) |
| 1 – 5t = -9 + 5s | so | 5t + 5s = 10 | (3) |

Solving for t and s in equations (1) and (2): 2t = -6 (1) – (2)

t = -3

Substituting this into equation (2): -3 + 4s = 5

4s = 8

s = 2

We now check to see whether these values for s and t work in equation (3):

$$5t + 5s = -15 + 10 = -5 \neq 10$$

So the lines do not intersect. So the lines are skew (since they do not meet and are not parallel).

4 (i)

$$\int \frac{1}{(1+x^2)^2}\,dx = \int \frac{1}{(1+\tan^2\theta)^2}\,dx \qquad x=\tan\theta \Rightarrow \frac{dx}{d\theta}=\sec^2\theta \Rightarrow dx = \sec^2\theta\, d\theta$$

$$= \int \frac{1}{(1+\tan^2\theta)^2}\sec^2\theta\, d\theta$$

But $1+\tan^2\theta = \sec^2\theta$

So, the integral becomes: $\int \frac{1}{(\sec^2\theta)^2}\sec^2\theta\, d\theta = \int \frac{1}{\sec^2\theta}\,d\theta = \int \cos^2\theta\, d\theta$ (as required)

(ii) $\int \cos^2 d\theta = \int \tfrac{1}{2}(1+\cos 2\theta)\,d\theta$ (**note**: you should <u>learn</u> the result $\cos^2\theta = \tfrac{1}{2}(1+\cos 2\theta)$)

So $\int \cos^2 d\theta = \tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta$

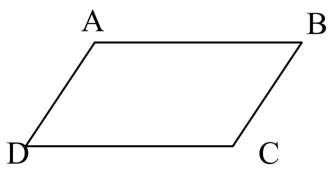
$$\int_0^1 \frac{1}{(1+x^2)^2}\,dx = \int_{x=0}^{1} \cos^2\theta\, d\theta = \int_{\theta=0}^{\frac{\pi}{4}} \cos^2\theta\, d\theta \qquad \begin{array}{l} x=0 \Rightarrow \theta = 0 \\ x=1 \Rightarrow \theta=\frac{\pi}{4}\end{array}$$

$$= \left[\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{4}}$$

$$= \left(\tfrac{\pi}{8}+\tfrac{1}{4}\right) - 0$$

$$= \tfrac{\pi}{8}+\tfrac{1}{4}$$

5




The vertices of the parallelogram are labelled in alphabetical order either in a clockwise or anti-clockwise direction.

Because ABCD is a parallelogram:

$$\overrightarrow{AD} = \overrightarrow{BC} = \mathbf{c}-\mathbf{b} = \begin{pmatrix}1\\-1\\-2\end{pmatrix}-\begin{pmatrix}3\\-2\\0\end{pmatrix}=\begin{pmatrix}-2\\1\\-2\end{pmatrix}$$

So the position vector of D is: $\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

(ii) Angle ABC is the angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{CB}$.

$$\overrightarrow{AB} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix} \quad \text{and} \quad \overrightarrow{CB} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

The scalar product of these vectors is: $\begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix} . \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = 2 + 3 - 6 = -1$

The magnitude of the two vectors are: $\left| \begin{pmatrix} 1 \\ -3 \\ -3 \end{pmatrix} \right| = \sqrt{1+9+9} = \sqrt{19}$ and $\left| \begin{matrix} 2 \\ -1 \\ 2 \end{matrix} \right| = \sqrt{4+1+4} = 3$.

The angle can be found using the formula: $\mathbf{a}.\mathbf{b} = ab\cos\theta$

Substituting into this formula gives: $1 \quad \sqrt{19} \quad 3 \quad \cos\theta$

So cosθ = -0.07647

i.e. θ = 94° (to the nearest degree).

6 (i) Implicit differentiation:

$$\frac{d}{dx}(xy^2) = 1y^2 + x2y\frac{dy}{dx} = y^2 + 2xy\frac{dy}{dx} \quad \text{(using the product rule).}$$

$$\frac{d}{dx}(2x) = 2$$

$$\frac{d}{dx}(3y) = 3\frac{dy}{dx}$$

So

$$y^2 + 2xy\frac{dy}{dx} = 2 + 3\frac{dy}{dx}$$

Putting all the $\frac{dy}{dx}$ terms together on the left hand side:

$$(2xy - 3)\frac{dy}{dx} = 2 - y^2$$

Therefore: $\frac{dy}{dx} = \frac{2 - y^2}{2xy - 3}$

(ii) Tangents parallel to the y-axis have infinite gradient.

$\frac{dy}{dx} = \infty$ if the denominator is zero, i.e. if $2xy \quad 3 \quad 0$ i.e. if $y = \frac{3}{2x}$.

If we substitute this expression for y into the equation of the curve, we get:

$$x\left(\frac{3}{2x}\right)^2 = 2x + \frac{9}{2x}$$

$$x\left(\frac{9}{4x^2}\right) = 2x + \frac{9}{2x}$$

$$\left(\frac{9}{4x}\right) = 2x + \frac{9}{2x}.$$

If we multiply both sides by 4x to remove the fractions we get:

$$9 = 8x^2 + 18$$
$$-9 = 8x^2$$

This is impossible, because $x^2$ cannot be negative. So there are no tangents parallel to the y-axis.

7 (i) $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-1/t^2}{2t} = \frac{-1}{2t^3}$$

(ii) At the point (4, -½), the value of $t$ is -2.

So: $\dfrac{dy}{dx} = \dfrac{-1}{2(-2)^3} = \dfrac{-1}{-16} = \dfrac{1}{16}$.

The equation of a tangent is $y = mx + c$, i.e. $y = \frac{1}{16}x + c$

Substitute in $x = 4$ and $y = -½$: $\frac{-1}{2} = \frac{1}{16}(4) + c \Rightarrow c = -\frac{3}{4}$

Therefore the equation of the tangent is $y = \frac{1}{16}x - \frac{3}{4}$ or $16y = x - 12$.

This rearranges to give: $x - 16y = 12$ (as required).

(iii) Put $x = t^2$, $y = \dfrac{1}{t}$ into $x - 16y = 12$ :-

$$t^2 - \frac{16}{t} = 12.$$

Multiply through by $t$ to remove the fraction: $t^3 - 16 = 12t$ or $t^3 - 12t - 16 = 0$.

We now need to solve this equation. We know t = -2 is one solution as that is one point where the tangent meets the curve. So $t + 2$ must be a factor.

We can therefore divide by $t + 2$ in order to find the other factors:

$$\begin{array}{r|l} & t^2 - 2t - 8 \\ t+2 & t^3 + 0t^2 - 12t - 16 \\ & \underline{t^3 + 2t^2} \\ & \quad -2t^2 - 12t \\ & \quad \underline{-2t^2 - 4t} \\ & \qquad -8t - 16 \\ & \qquad \underline{-8t - 16} \\ & \qquad\quad 0 \end{array}$$

So we need to solve $t^2 - 2t - 8 = 0$. This factorises: $(t - 4)(t + 2) = 0$.

So the other value of the parameter is $t = 4$.

8 (i) $$\frac{3x+4}{(1+x)(2+x)^2} = \frac{A}{1+x} + \frac{B}{2+x} + \frac{C}{(2+x)^2}$$

$$= \frac{A(2+x)^2 + B(1+x)(2+x) + C(1+x)}{(1+x)(2+x)^2}$$

Therefore: $3x+4 = A(2+x)^2 + B(1+x)(2+x) + C(1+x)$

Substitute $x$ = -2: -2 = -C i.e. C = 2
Substitute $x$ = -1: 1 = A i.e. A = 1.

Now substitute any further value for $x$:
E.g. $x = 0$: 4 = 4A + 2B + C
4 = 4 + 2B + 2
2B = -2, i.e. B = -1

Therefore $$\frac{3x+4}{(1+x)(2+x)^2} = \frac{1}{1+x} - \frac{1}{2+x} + \frac{2}{(2+x)^2}$$

(ii) $$\frac{3x+4}{(1+x)(2+x)^2} = \frac{1}{1+x} - \frac{1}{2+x} + \frac{2}{(2+x)^2} = (1+x)^{-1} - (2+x)^{-1} + 2(2+x)^{-2}.$$

We use the Binomial expansion formula: $(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2$.

$$(1+x)^{-1} = 1 - x + \frac{(-1)(-2)}{2}x^2 = 1 - x + x^2 - \ldots$$ (This is valid if -1 < $x$ < 1).

$$(2+x)^{-1} = 2(1+\tfrac{x}{2})^{-1} = 2^{-1}(1+\tfrac{x}{2})^{-1}$$
$$= \tfrac{1}{2}\left(1 + (-1)\tfrac{x}{2} + \frac{(-1)(-2)}{2}(\tfrac{x}{2})^2 + \ldots\right)$$
$$= \tfrac{1}{2}\ 1 - \tfrac{x}{2} + \tfrac{x^2}{4} + \ldots$$
$$= \tfrac{1}{2} - \tfrac{x}{4} + \tfrac{x^2}{8}$$

(This is valid if -1 < $\tfrac{x}{2}$ < 1,
i.e. if -2 < $x$ < 2)

$$(2+x)^{-2} = 2^{-2}(1+\tfrac{x}{2})^{-2} = \tfrac{1}{4}(1 + (-2)\tfrac{x}{2} + \frac{(-2)(-3)}{2}\ \tfrac{x}{2}^{\ 2} + \ldots)$$
$$= \tfrac{1}{4}(1 - x + \tfrac{3x^2}{4} + \ldots)$$
$$= \tfrac{1}{4} - \tfrac{x}{4} + \tfrac{3x^2}{16} + \ldots$$

(This is also valid if
-2 < $x$ < 2)

So, $$\frac{3x+4}{(1+x)(2+x)^2} = (1+x)^{-1} - (2+x)^{-1} + 2(2+x)^{-2} = 1 - x + x^2 - (\tfrac{1}{2} - \tfrac{x}{4} + \tfrac{x^2}{8}) + 2(\tfrac{1}{4} - \tfrac{x}{4} + \tfrac{3x^2}{16})$$
$$= 1 - \tfrac{5}{4}x + \tfrac{5}{4}x^2$$

(iii) The expansion is valid if -1 < $x$ < 1.

9 (i)

$$\frac{d\theta}{dt} = -k\ (\theta - 20)$$

This is the rate at which the temperature of the object changes.

k is the proportionality constant. It is negative as the object is cooling.

This is the difference in temperature between the object and the surroundings.

(ii) We separate out the variables: $\frac{1}{\theta - 20} d\theta = -kdt$

Put in integral signs: $\int \frac{1}{\theta - 20} d\theta = -\int kdt$

So $\ln(\theta - 20) = -kt + c$

i.e. $\theta - 20 = e^{-kt+c} = e^{-kt}e^{c}$

Therefore: $\theta = 20 + Ae^{-kt}$ (where $A = e^{c}$ )

When $t = 0$, $\theta = 100$: So $100 = 20 + Ae^{0} \Rightarrow A = 80$

So $\theta = 20 + 80e^{-kt}$

When $t = 5$, $\theta = 68$: So $68 = 20 + 80e^{-k\times 5}$

$\Rightarrow 48 = 80e^{-k\times 5}$

$\Rightarrow \frac{48}{80} = e^{-5k}$

$\Rightarrow \frac{3}{5} = e^{-5k}$

Take logarithms: $\ln(\frac{3}{5}) = -5k$

i.e. $k = \frac{-1}{5}\ln\frac{3}{5} = \frac{1}{5}\ln\frac{5}{3}$ (as $\ln\frac{5}{3} = -\ln\frac{3}{5}$ )

So we get: $\theta = 20 + 80e^{-\frac{1}{5}\ln(\frac{5}{3})\,t}$ (as required),

(iii) If the liquid falls by another 32°C, the new temperature will be 36°C:

$36 = 20 + 80e^{-\frac{1}{5}\ln(\frac{5}{3})\,t}$

$16 = 80e^{-\frac{1}{5}\ln(\frac{5}{3})\,t}$

i.e. $0.2 = e^{-\frac{1}{5}\ln(\frac{5}{3})\,t}$

$\ln 0.2 = -\frac{1}{5}\ln(\frac{5}{3})\ t$

So, t = 15.75 minutes.

So it cools by a further 32 °C after another 10.75 minutes.