C4 Jan 2010

$$\begin{array}{r} x^2+6x-4 \\ x^2+5x+2\overline{\big)\,x^4+11x^3+28x^2+3x+1} \end{array}$$ Quotient

1)

$$\begin{array}{rrrr} x^4+5x^3 & +2x^2 & & \\ \hline 6x^3 & +26x^2 & +3x & +1 \\ 6x^3 & +30x^2 & +12x & \\ \hline & -4x^2 & -9x & +1 \\ & -4x^2 & -20x & -8 \\ \hline & & 11x & +9 \end{array}$$ Remainder

2) OA = – 5i – 10j + 12k, OB = i + 2j – 3k, OC = 3i + 6j + pk

AB = OB – OA = 6i + 12j – 15k BC = OC – OA = 2i + 4j + (p+3)k

i) $\angle ABC = 90° \Rightarrow AB \bullet BC = 0 \Rightarrow 2\times 6 + 12\times 4 - 15(p+3) = 0 \Rightarrow p = 1$

ii) ABC straight line implies AB and BC are parallel.

AB = 3 x BC therefore 3 x (p + 3) = -15 p = -8

3)

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos 2x}{\cos^2 x}dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{2\cos^2 x - 1}{\cos^2 x}dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 2 - \sec^2 x dx = \left[2x - \tan x\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \left(2\times\frac{\pi}{3} - \tan\frac{\pi}{3}\right) - \left(2\times\frac{\pi}{4} - \tan\frac{\pi}{4}\right) = \frac{2\pi}{3} - \sqrt{3} - \frac{\pi}{2} + 1 = \frac{\pi}{6} - \sqrt{3} + 1$$

4) $\int_1^e \frac{1}{t(2+\ln t)^2}dt$ u = 2 + lnt dt = tdu, t = 1 u = 2, t = e u = 3

$$= \int_2^3 \frac{1}{tu^2}tdu = \int_2^3 \frac{1}{u^2}du = \left[-\frac{1}{u}\right]_2^3 = -\frac{1}{3} - -\frac{1}{2} = \frac{1}{6}$$

5) i) $(1+x)^{\frac{1}{3}} = 1 + \frac{1}{3}x + \frac{\frac{1}{3}\times\frac{-2}{3}}{2!}x^2 = 1 + \frac{1}{3}x - \frac{1}{9}x^2$

ii) a) $(8+16x)^{\frac{1}{3}} = (8)^{\frac{1}{3}} \times (1+2x)^{\frac{1}{3}} = 2\left\{1 + \frac{1}{3}\times 2x - \frac{1}{9}\times(2x)^2\right\} = 2 + \frac{4}{3}x - \frac{8}{9}x^2$

b) $-\frac{1}{2} \le x \le \frac{1}{2}$

6) x = 9t – ln (9t) y = t^3 – ln (t^3)

$$\frac{dx}{dt} = 9 - \frac{9}{9t} = \frac{9t-1}{t} \qquad \frac{dy}{dt} = 3t^2 - \frac{3t^2}{t^3} = \frac{3t^3-3}{t}$$

$$\frac{dy}{dx} = \left(\frac{3t^3-3}{t}\right)\times\left(\frac{t}{9t-1}\right) = \frac{3t^3-3}{9t-1} = \frac{3(t^3-1)}{9t-1}$$

$$\frac{dy}{dx} = \frac{3(t^3-1)}{9t-1} = 3 \qquad \begin{array}{l} t^3 - 1 = 9t - 1 \\ t^3 - 9t = 0 \end{array}$$

$t(t^2 – 9) = 0$ therefore t = 0 or t^2 = 9 therefore t = 0, 3 or -3. ln(-27) and ln(0) are not defined therefore t = -3 and t = 0 are not valid which means t = 3 is the only solution.

7) $x^3 + 2x^2y = y^3 + 15$

$$3x^2dx + 2(2xydx - x^2dy) = 3y^2dy$$

$$(3x^2 + 4xy)dx = (3y^2 - 2x^2)dy$$

$$\frac{dy}{dx} = \frac{3x^2 + 4xy}{3y^2 - 2x^2} = \frac{3\times 2^2 + 4\times 2\times 1}{3\times 1^2 - 2\times 2^2} = -4$$

x = 2, y = 1 and m = $\frac{1}{4}$

$y - 1 = \frac{1}{4}(x - 2)$ 0 = x – 4y + 2

8) i) -sinx $e^{\cos x}$

ii)

$$\int_0^{\frac{\pi}{2}} \cos x \sin x e^{\cos x} dx = \left[-\cos x e^{\cos x}\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x e^{\cos x} dx = \left[-\cos x e^{\cos x} + e^{\cos x}\right]_0^{\frac{\pi}{2}}$$

$$= \left(-\cos\frac{\pi}{2} e^{\cos\frac{\pi}{2}} + e^{\cos\frac{\pi}{2}}\right) - \left(-\cos 0 e^{\cos 0} + e^{\cos 0}\right) = 1$$

9) r = (3 + t)i + (1 – t)j + (1 + 2t)k

i) t = 1 OP = 4i + 3k

$$\cos\theta = \frac{4\times 1 + 0\times -1 + 3\times 2}{\sqrt{4^2 + 0^2 + 3^2} \times \sqrt{1^2 + (-1)^2 + 2^2}} = \frac{10}{5\sqrt{6}}$$

$$\theta = 35.3°$$

ii) OQ = (3 + t)i + (1 – t)j + (1 + 2t)k $OQ \bullet (i - j + 2k) = 0$

(3 + t) – (1 – t) + 2(1 + 2t) = 0 t = $-\frac{2}{3}$

OQ = $\begin{pmatrix} 2\frac{1}{3} \\ 1\frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$

iii) $\sqrt{\left(3 - \frac{2}{3}\right)^2 + \left(1 + \frac{2}{3}\right)^2 + \left(1 + 2\times -\frac{2}{3}\right)^2} = \sqrt{\frac{25}{3}} = \frac{5\sqrt{3}}{3} = 2.89$

10) i) $\frac{1}{(3-x)(6-x)} = \frac{A}{3-x} + \frac{B}{6-x} = \frac{1}{3(3-x)} - \frac{1}{3(6-x)}$

$$1 = A(6-x) + B(3-x)$$

x = 3 implies A = 1/3,
x = 6 implies B = -1/3

$$\frac{dx}{dt} = k(3-x)(6-x)$$

$$\int \frac{1}{(3-x)(6-x)}dx = \int kdt$$

$$\frac{1}{3}\int \frac{1}{3-x} - \frac{1}{6-x}dx = \int kdt$$

$$-\ln(3-x) + \ln(6-x) = 3kt + c$$

$$\ln\left(\frac{6-x}{3-x}\right) = 3kt + c$$

ii) a)

x = 0   t = 0

x = 1   t = 1

ln(2) = c

$$\ln\left(\frac{5}{2}\right) = 3k + \ln 2$$

$$\ln\left(\frac{5}{2}\right) - \ln 2 = 3k$$

$$k = \frac{1}{3}\ln\left(\frac{5}{4}\right)$$

b) t = 2

$$\ln\left(\frac{6-x}{3-x}\right) = 3\times\frac{1}{3}\ln\left(\frac{5}{4}\right)\times 2 + \ln 2 = 2\ln\left(\frac{5}{4}\right) + \ln 2 = \ln\left(\frac{25}{8}\right)$$

$$\frac{6-x}{3-x} = \frac{25}{8}$$

$$48 - 8x = 75 - 25x$$

$$17x = 27$$

$$x = \frac{27}{17} = 1.59$$