CORE 4 Jan 2009

1. $\dfrac{20-5x}{6x^2-24x} = \dfrac{5(4-x)}{6x(x-4)} = \underline{\underline{-\dfrac{5}{6x}}}$

2. $\displaystyle\int \underset{u}{x}\,\underset{v'}{\sec^2 x}\,dx = x\tan x - \int \tan x\,dx$

$u' = 1$

$v = \tan x$

$\displaystyle = x\tan x - \int \frac{\sin x}{\cos x}\,dx$

$= \underline{\underline{x\tan x + \ln|\cos x| + c}}$

3. (i) $(1+2x)^{\frac{1}{2}} = 1 + \left(\frac{1}{2}\right)(2x) + \underbrace{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)}_{2}(2x)^2 + \underbrace{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}_{6}(2x)^3$

$= \underline{\underline{1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3}}$

(ii) $(1+x)^{-3} = 1 + (-3)x + \underbrace{(-3)(-4)}_{2}x^2 + \underbrace{(-3)(-4)(-5)}_{6}x^3$

$= 1 - 3x + 6x^2 - 10x^3$

$(1+2x)^{\frac{1}{2}}(1+x)^{-3} \simeq \left(1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3\right)\left(1 - 3x + 6x^2 - 10x^3\right)$

$= 1 - 3x + 6x^2 - 10x^3$

$\quad x - 3x^2 + 6x^3$

$\quad -\frac{1}{2}x^2 + \frac{3}{2}x^3$

$\quad + \frac{1}{2}x^3$

$= \underline{\underline{1 - 2x + \frac{5}{2}x^2 - 2x^3}}$

(iii) $|2x| < 1 \qquad \underline{\underline{|x| < \frac{1}{2}}}$

4. $\displaystyle\int_0^{\pi/4} (1+\sin x)^2\,dx = \int_0^{\pi/4} (1 + 2\sin x + \sin^2 x)\,dx$

$\displaystyle = \int_0^{\pi/4} \left(1 + 2\sin x + \frac{1}{2} - \frac{1}{2}\cos 2x\right)dx$

$= \left[\frac{3x}{2} - 2\cos x - \frac{1}{4}\sin 2x\right]_0^{\pi/4}$

$= \left[\frac{3\pi}{8} - \sqrt{2} - \frac{1}{4}\right] - \left[0 - 2 - 0\right]$

$= \underline{\underline{\frac{3\pi}{8} - \sqrt{2} + \frac{7}{4}}}$

5. (i) $u = \sqrt{x}$, $u^2 = x$, $2u\,du = dx$

$$\int \frac{1}{x(1+\sqrt{x})}\,dx = \int \frac{2u}{u^2(1+u)}\,du$$

$$= \underline{\underline{\int \frac{2}{u(1+u)}\,du}}$$

$x = 9$, $u = 3$
$x = 1$, $u = 1$

$$\int_1^3 \frac{2}{u(1+u)}\,du$$

$$\frac{2}{u(1+u)} = \frac{A}{u} + \frac{B}{1+u}$$

$2 = A(1+u) + Bu$

$u = 0 \rightarrow A = 2$, $u = -1 \rightarrow B = -2$

$$\int_1^3 \frac{2}{u} - \frac{2}{1+u}\,du$$

$$= 2\Big[\ln u - \ln(1+u)\Big]_1^3$$

$$= 2\left[\ln\left(\frac{u}{1+u}\right)\right]_1^3 = 2\left[\ln \tfrac{3}{4} - \ln\tfrac{1}{2}\right]$$

$$= 2\ln \tfrac{3}{2} = \underline{\underline{\ln\tfrac{9}{4}}}$$

6. (i) meets x axis when $y = 0$

$0 = t - 3$, $t = 3$

$x = t^2 - 6t + 4$
$= 9 - 18 + 4 = -5$

$\underline{\underline{\text{coords } (-5, 0)}}$

(ii) $x = (y+3)^2 - 6(y+3) + 4$
$x = y^2 + 6y + 9 - 6y - 18 + 4$
$\underline{\underline{x = y^2 - 5}}$

(iii)

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= 1 \times \frac{1}{2t-6} = \frac{1}{2t-6}$$

$t = 2 \Rightarrow \frac{dy}{dx} = -\frac{1}{2}$

when $t = 2$ $x = 4 - 12 + 4 = -4$
$y = -1$

tangent $y + 1 = -\frac{1}{2}(x+4)$

$2y + 2 = -x - 4$

$\underline{\underline{x + 2y + 6 = 0}}$

$\left\{ \begin{array}{l} \text{OR } x = y^2 - 5 \\ \frac{dx}{dy} = 2y, \ \frac{dy}{dx} = \frac{1}{2y} = -\frac{1}{2} \end{array} \right.$ then as above

7. (i) $\underline{r} = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} + t\begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$

$$\begin{pmatrix} 9 \\ 7 \\ 5 \end{pmatrix} - \begin{pmatrix} 7 \\ 8 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

$$\underline{r}_2 = \begin{pmatrix} 9 \\ 7 \\ 5 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} + t\begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$$

① $9 + 2\lambda = 2 + t$

② $7 - \lambda = -3 + 4t \quad \Rightarrow \lambda = 10 - 4t$

③ $5 + 3\lambda = 5 - 2t$ sub in ①

① $9 + 2(10 - 4t) = 2 + t$

$9 + 20 - 8t = 2 + t$

$27 = 9t$

$t = 3$

when $t = 3$, $\lambda = -2$

check consistent ① $9 + 2(-2) = 2 + 3$ ✓ $= 5$

② $7 - (-2) = -3 + 4(3)$ ✓ $= 9$

③ $5 + 3(-2) = 5 - 2(3)$ ✓ $= -1$

Hence lines intersect at $(5, 9, -1)$

(ii) $$\cos\theta = \frac{\begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}}{\sqrt{1^2 + 4^2 + (-2)^2}\sqrt{2^2 + (-1)^2 + 3^2}}$$

$$= \frac{2 - 4 - 6}{\sqrt{21}\sqrt{14}} = \frac{-8}{\sqrt{21}\sqrt{14}}$$

$\theta = 117.8^\circ$

Hence acute angle $= 180 - 117.8 = 62.2^\circ$ OR 1.09^c

8. (i) $x^3 + y^3 = 6xy$

$$3x^2 + 3y^2\frac{dy}{dx} = 6y + 6x\frac{dy}{dx}$$

$$x^2 + y^2\frac{dy}{dx} = 2y + 2x\frac{dy}{dx}$$

$$\frac{dy}{dx}(y^2 - 2x) = 2y - x^2$$

$$\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}$$

8 (ii) $x^3 = 2^4 = 16$, $y^3 = 2^5 = 32$

So $x^3 + y^3 = 48 = 6 \times 2^{4/3} \times 2^{5/3} = 6 \times 8$ ✓

$$\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x} = \frac{2 \times 2^{5/3} - 2^{8/3}}{2^{10/3} - 2 . 2^{4/3}} = \frac{2^{8/3} - 2^{8/3}}{2^{10/3} - 2^{7/3}} = \underline{\underline{0}}$$

(iii) $a^3 + a^3 = 6a^2$

$2a^3 = 6a^2$

$a^3 - 3a^2 = 0$

$a^2(a - 3) = 0$, $\underline{\underline{a = 3}}$

$\underset{\sim}{a \neq 0}$

$$\text{Gradient} = \frac{2y - x^2}{y^2 - 2x} = \frac{6 - 9}{9 - 6} = \underline{\underline{-1}}$$

9. (i) $\underline{\underline{\dfrac{d\theta}{dt} = k(160 - \theta)}}$

(ii) $\int k \, dt = \int \dfrac{1}{160 - \theta} \, d\theta$

$kt = -\ln(160 - \theta) + c$

$t = 0, \theta = 20$

$0 = -\ln 140 + c$, $c = \ln 140$

$t = 5, \theta = 65$ $\quad 5k = -\ln 95 + \ln 140$

$k = \dfrac{1}{5} \ln \dfrac{140}{95} = \dfrac{1}{5} \ln \dfrac{28}{19}$

$t = 10$, $\quad \left(\dfrac{1}{5} \ln \dfrac{28}{19}\right) \times 10 = -\ln(160 - \theta) + \ln 140$

$\ln(160 - \theta) = \ln 140 - 2 \ln \dfrac{28}{19}$

$\ln(160 - \theta) = \ln\left(\dfrac{140}{\left[28/19\right]^2}\right)$

$160 - \theta \simeq 64.46$

$\underline{\underline{\theta \simeq 96^\circ}}$