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A2 GCE MATHEMATICS (MEI)

4754/01B Applications of Advanced Mathematics (C4) Paper B: Comprehension

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Day-Night Maps

On many inter-continental flights you will see a *day-night map* displayed, like that in Fig. 1. It shows those parts of the earth that are in daylight and those that are in darkness. Such maps usually show the position of the aeroplane. They also, as in this case, often show the point that is immediately under the sun; at that point the sun is directly overhead.

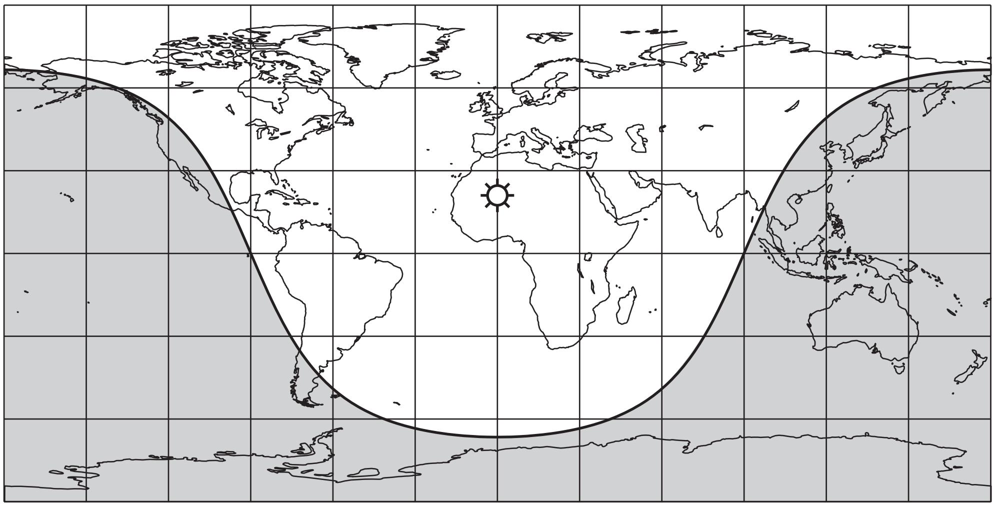

Fig. 1

Fig. 1 shows the day-night map when it is mid-day in the United Kingdom on mid-summer's day in the northern hemisphere.

Modelling assumptions

Fig. 2 illustrates the earth as a 3-dimensional object being illuminated by the sun. At any time the sun is shining on approximately half of the earth's surface but not on the other half. The two regions are separated by a circle on the earth's surface called the *terminator*. This is represented in Fig. 1 as the curve separating the light and dark regions.

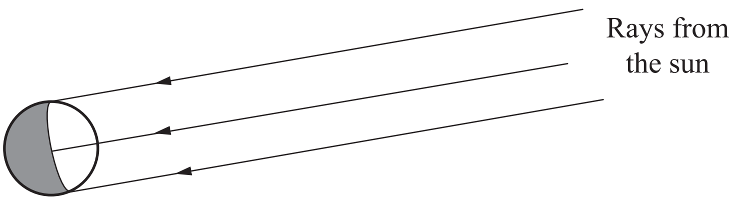

Fig. 2

In this article, a number of modelling assumptions are made to simplify the situation.

- The sun is taken to be a point so that at any time it is either above the horizon or below it, but never partly above and partly below.
- All light rays coming from the sun to the earth are parallel.
- The effects of refraction (bending of the light by the earth's atmosphere) are negligible.
- The earth is a perfect sphere.

The effects of these assumptions are that, at any time, exactly (and not approximately) half of the earth's surface is being illuminated by the sun and there is no twilight; at any place, it is either day or night. None of the assumptions is actually quite true, but they are all close enough to provide a good working model.

Fig. 3, below, is in two dimensions; it shows a section of the earth through its centre, O, in the same plane as the sun.

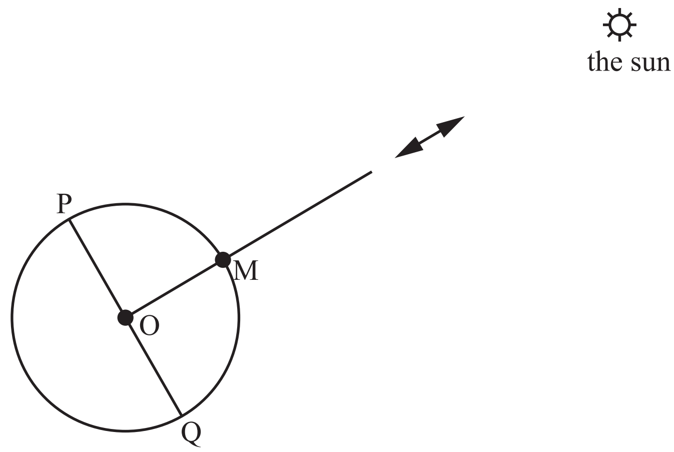

Fig. 3

The line from the centre of the earth to the sun cuts the surface of the earth at the *sub-solar point*, M. If you were standing at M you would see the sun directly overhead. On a typical day-night map there is a picture of the sun at the sub-solar point.

The points P and Q are on the terminator: $\angle POM = \angle QOM = 90°$. If you were standing at P or Q it would either be the moment of sunrise or the moment of sunset for you.

The cartesian equation of the terminator

Fig. 4 shows the map of the world with x- and y-axes superimposed.

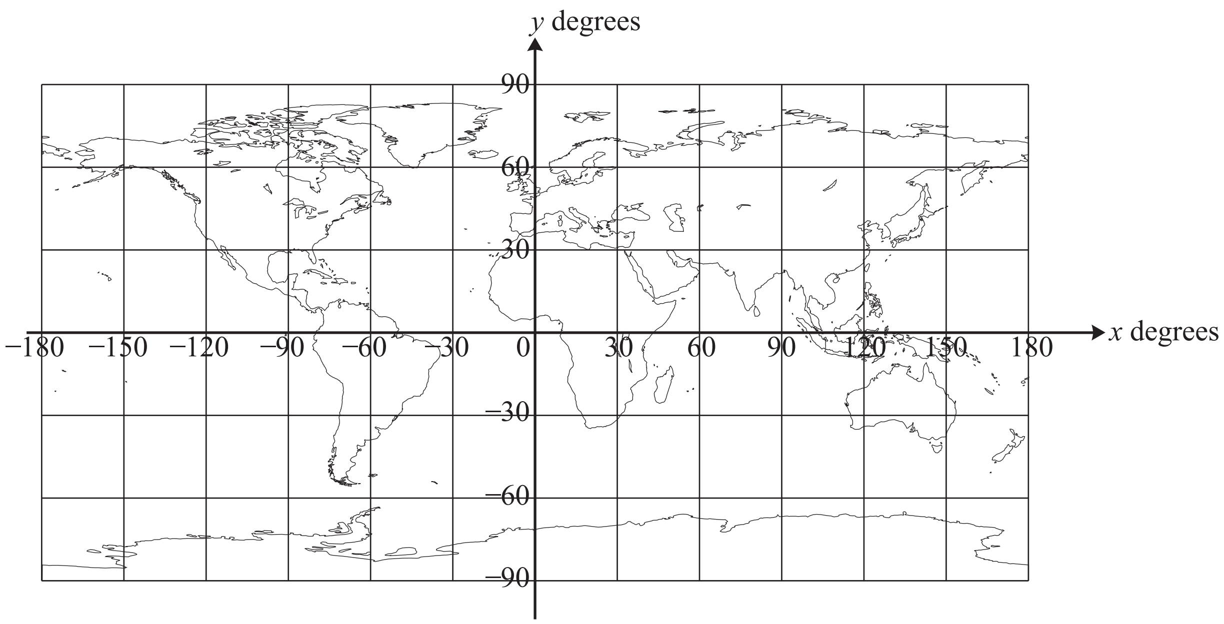

Fig. 4

On these axes:

- x represents longitude, going from $-180°$ ($180°$ west) to $+180°$ ($180°$ east),
- y represents latitude, going from $-90°$ ($90°$ south) to $+90°$ ($90°$ north).

The lines parallel to the x-axis are called lines of latitude. On the earth's surface they are actually circles. The x-axis itself is the equator.

The lines parallel to the y-axis are called lines of longitude, or *meridians*. Each meridian is actually a semicircle along the earth's surface joining the north pole and the south pole. The y-axis is the zero meridian; it passes through Greenwich in London and so is called the Greenwich meridian.

The y-coordinate of the sub-solar point is its latitude, measured in degrees, and is called the *declination* of the sun. In this article the declination of the sun is denoted by α. During a year, the value of α varies between $+23.44°$ on mid-summer's day in the northern hemisphere and $-23.44°$ on mid-winter's day. Fig. 1 shows the situation on mid-summer's day, when the sun is at its most northerly, and so $\alpha = 23.44°$.

Using these axes, it is possible to show that the equation of the terminator on the map, for the time and day shown in Fig. 1, can be written as the cartesian equation

$$\tan y = -2.306 \cos x. \qquad (1)$$

This can also be written as

$$y = \arctan(-2.306 \cos x).$$

Fig. 5 shows this curve.

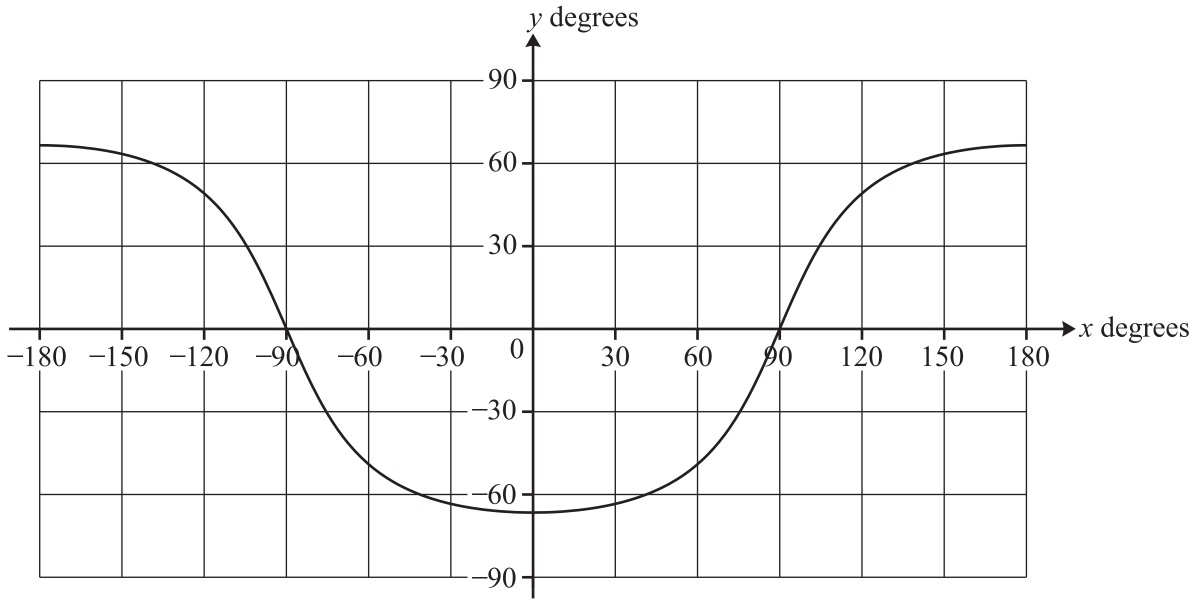

Fig. 5

When this curve is superimposed on the map of the world, and the correct region is shaded, the day-night map in Fig. 1 is produced.

The terminator is a circle on the earth's surface and so it is quite surprising that the curve in Fig. 5 looks nothing like a circle. There are two points to be made.

- The map in Fig. 4 is formed from a cylinder that has been cut along the line $x = \pm 180$ and laid flat. $x = +180$ and $x = -180$ are the same line. So the curve is continuous.
- The earth is a sphere and representing it on a cylinder causes distortion; this affects the shape of the curve on the map. In particular, the polar regions become very distorted, and, along with them, the circular shape of the terminator.

The representation used to draw a sphere on a flat sheet of paper is called the map's *projection*. There are very many different map projections; the one used for day-night maps is called *equirectangular cylindrical* (or *plate carrée*).

The terminator at other times and on other dates

So far, only one time and day of the year has been considered, mid-day on mid-summer's day in the northern hemisphere when the declination of the sun is $+23.44°$. What about other times of day? And other days of the year?

The answer to the question about different times of day is that, as the earth rotates, the sub-solar point moves along its circle of latitude and the terminator moves with it, keeping the same shape.

The question about other days of the year relates to the declination of the sun. On mid-summer's day, $\alpha = 23.44°$. Equation (1) is $\tan y = -2.306 \cos x$; the number 2.306 arises because

$$\frac{1}{\tan 23.44°} = 2.306.$$

For a general value of α, the number 2.306 is replaced by

$$\frac{1}{\tan \alpha}$$

and so the equation of the terminator can be written in general form as

$$\tan y = -\frac{1}{\tan \alpha} \cos x. \qquad (2)$$

Equation (2) makes it possible to draw a graph illustrating the terminator for any possible declination of the sun. Fig. 6 shows the terminator in the cases where the sun has declination 10° north, 1° north, 5° south and 15° south. In each case the time is mid-day on the Greenwich meridian.

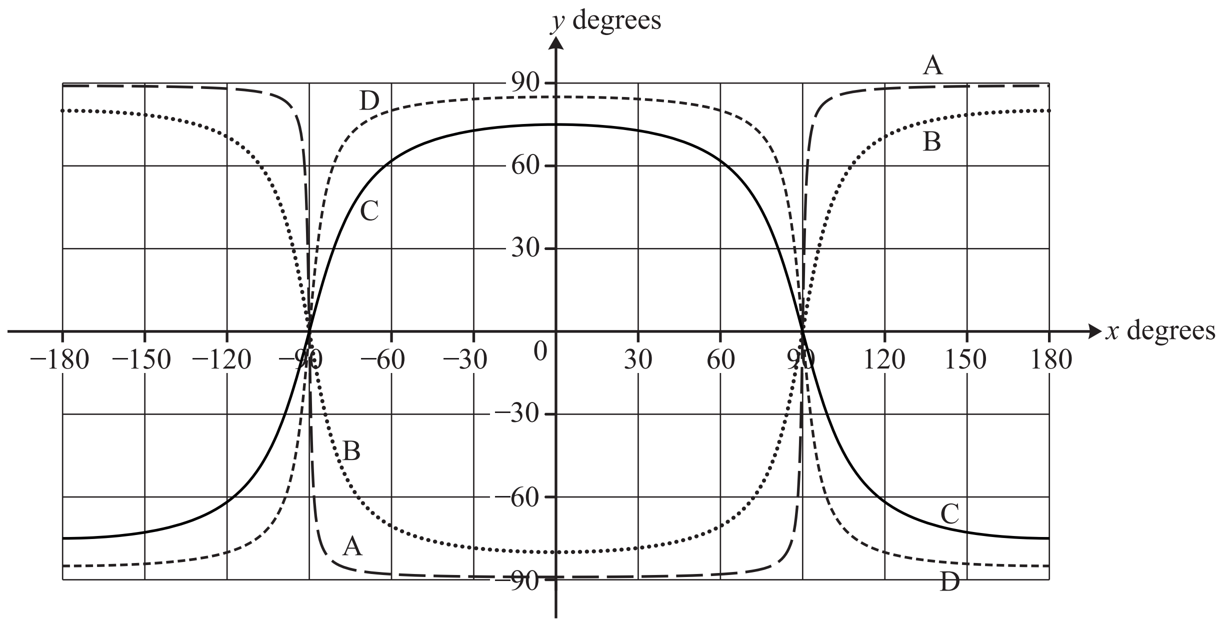

Fig. 6

The declination of the sun

In Fig. 1 the sun is at its most northerly point with declination +23.44°. On mid-winter's day its declination is −23.44°. To good approximation, the value of the declination follows a sine curve between these two values, as shown in Fig. 7. Day 1 is January 1st.

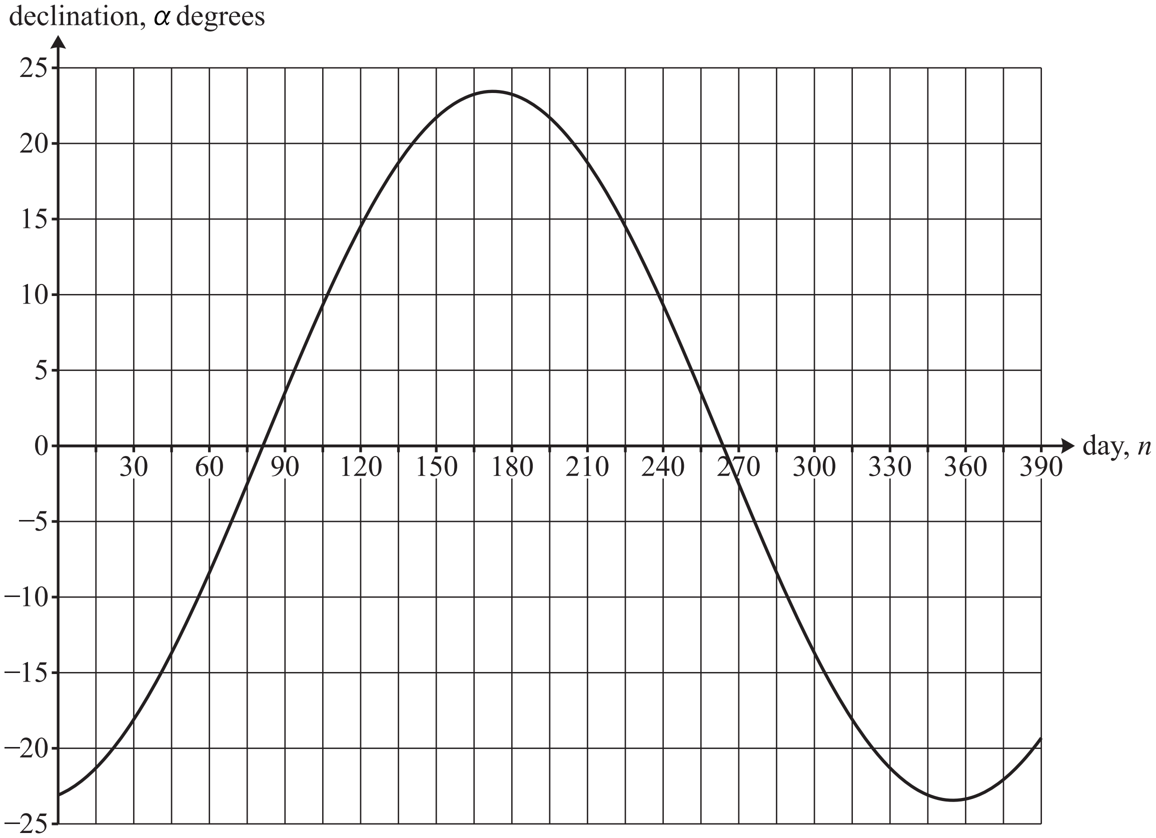

Fig. 7

The equation of the curve in Fig. 7 is

$$\alpha = -23.44 \times \cos\left(\frac{360}{365} \times (n + 10)\right). \qquad (3)$$

(This approximation is based on the modelling assumption that the orbit of the earth around the sun is a circle; it is actually an ellipse.)

Hours of daylight

If you travel north when it is summer in the northern hemisphere, you will notice that the days become longer and the nights shorter. The graph of the terminator allows you to see how this happens.

The earth rotates on its axis once every day. So it turns through 360° every 24 hours or 15° per hour. So every 15° of longitude (ie along the x-axis in Fig. 5) corresponds to 1 hour of time.

Fig. 8 is the same as Fig. 5 (the declination of the sun is +23.44°) but the horizontal axis represents the time difference from Greenwich, measured in hours, rather than longitude measured in degrees.

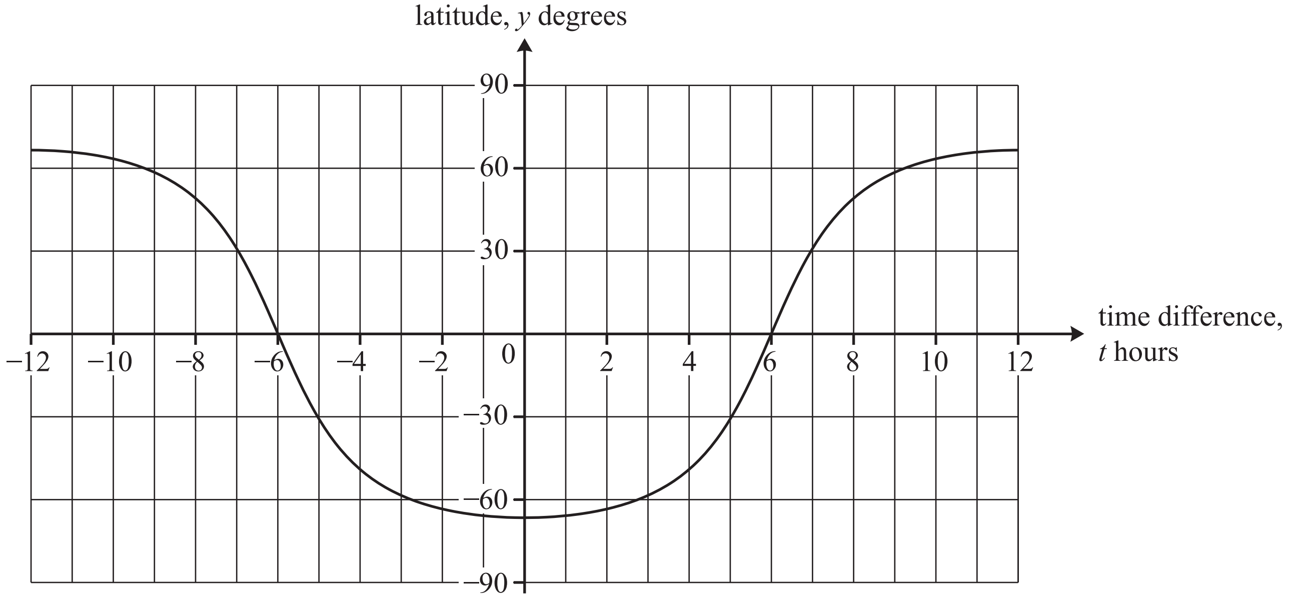

Fig. 8

The graph in Fig. 8 shows y against t. Since $x = 15t$, the equation of the terminator can be written as

$$\tan y = -\frac{1}{\tan\alpha}\cos(15t). \qquad (4)$$

Fig. 8 shows you that at latitude 60° north the terminator passes approximately through time +9 hours and –9 hours so that there are about 18 hours of daylight. Oslo has latitude 60° north.

You can also see that at latitude 30° north there are about 14 hours of daylight on this day of the year. Cairo has latitude 30° north.

So Oslo has about 4 more hours of daylight than Cairo on this day.

At the start of the article, it was stated that one effect of the modelling assumptions is to ignore twilight. This is the time when the sun is just below the horizon. The effect of twilight is particularly noticeable in places with high latitudes, for example Oslo, in the summer so that it is nearly light for even longer.