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Tuesday 17 January 2012 – Morning

A2 GCE MATHEMATICS (MEI)

4754B Applications of Advanced Mathematics (C4) Paper B: Comprehension

INSERT

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Duration: Up to 1 hour

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Curves of Constant Width

Introduction

Imagine you need to move a heavy object over level ground. You could rest your object on a horizontal platform and use cylindrical rollers, all with the same radius, as shown in Fig. 1.

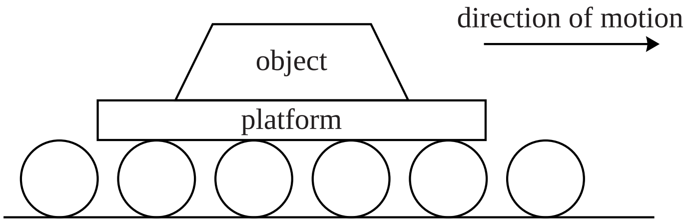

Fig. 1

Using cylindrical rollers in this way would ensure that the motion is smooth and horizontal; the object remains at the same height above the ground at all times.

If the cross-section of each roller was an ellipse, rather than a circle, then the motion would not be smooth. Fig. 2 shows different orientations of an elliptical roller.

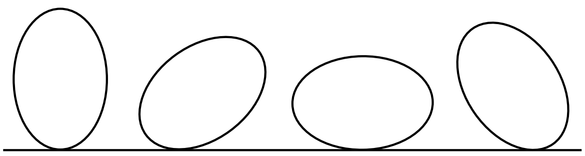

Fig. 2

You might be surprised to learn that there are infinitely many shapes which, like the circle, form the cross-section of a roller that would produce smooth horizontal motion of the object. These shapes are said to have *constant width* and the boundary of such a shape is called *a curve of constant width*. In this article you will be introduced to several types of curves of constant width.

Reuleaux triangle

The simplest non-circular curve of constant width, shown in Fig. 3, is named after Franz Reuleaux (1829–1905), a German mathematician and engineer. It is based on an equilateral triangle of side length l on which three circular arcs are drawn; each arc is centred on one vertex and passes through the other two vertices.

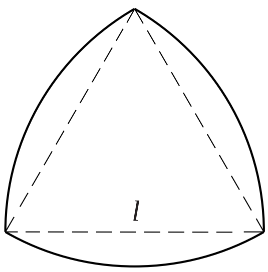

Fig. 3

If a roller with this cross-section rolled along a horizontal surface then the highest point would always be at a height l above the surface. This is illustrated in Fig. 4.

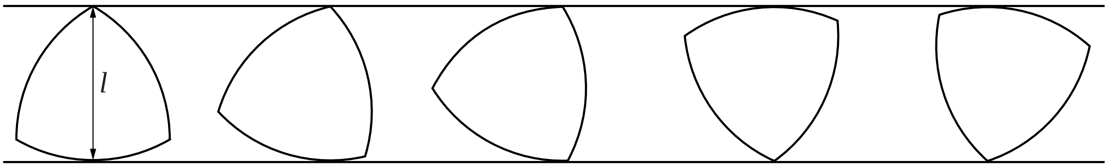

Fig. 4

In one revolution, assuming no slipping, the roller would move forward a distance πl, the same distance as for a cylindrical roller of diameter l.

In a similar way, arcs can be added to any regular polygon with an odd number of sides to make a curve of constant width. A Reuleaux pentagon, with constant width l, is shown in Fig. 5. In this curve, each arc is centred on the opposite vertex of the pentagon; for example, the arc CD has centre A and radius l.

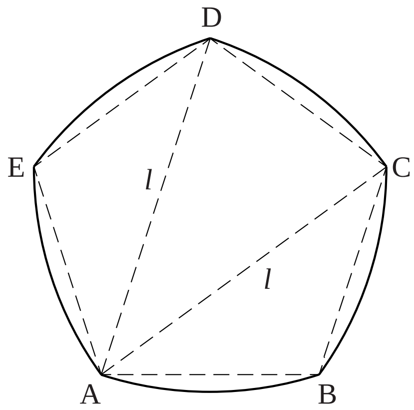

Fig. 5

Many mathematical properties of curves of constant width are known. Two of particular interest are given here.

- Every curve of constant width l has perimeter πl.
- Of all the curves of constant width l, the circle encloses the greatest area, $\frac{\pi}{4}l^2 \approx 0.785l^2$, and the Reuleaux triangle encloses the smallest area, $\left(\frac{\pi - \sqrt{3}}{2}\right) l^2 \approx 0.705l^2$.

Smooth curves of constant width

With the exception of the circle, the curves of constant width met so far have been created by constructing arcs on the sides of certain regular polygons. At each vertex of the regular polygon, two arcs meet but they do so in such a way that the curve is not smooth. For the Reuleaux pentagon in Fig. 6 there are two tangents at vertex A, one on each arc. The angle between these tangents is 144°.

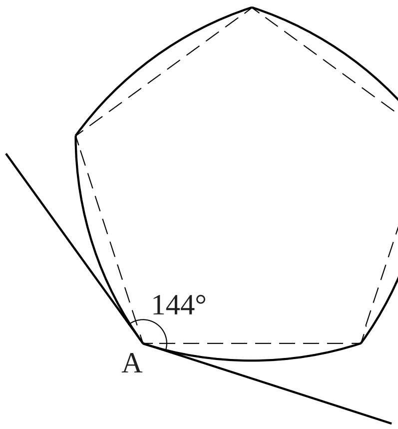

Fig. 6

It is possible to construct curves of constant width which are smooth at all points; one way of doing this is as follows.

- Draw an equilateral triangle ABC with side length R
- Extend the sides a distance r beyond each vertex, as shown in Fig. 7a
- Construct two circular arcs centred on A: arc A_1A_2 with radius r and arc B_2C_1 with radius $R + r$
- Construct similar arcs centred on B and C to give the curve shown in Fig. 7b.

This curve has constant width $R + 2r$ and is smooth. Every point on the curve has a unique tangent and the distance between parallel tangents is constant.

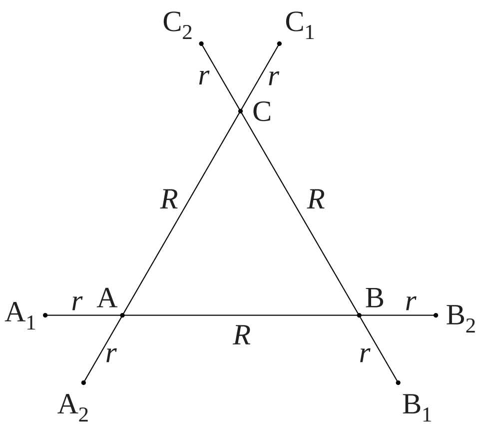

Fig. 7a

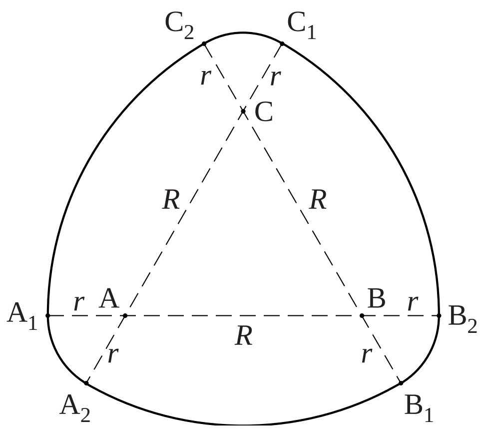

Fig. 7b

This method can be used on other regular polygons with an odd number of sides. One example of this is shown in Fig. 8. Notice that it is the longest diagonals of the heptagon that are extended.

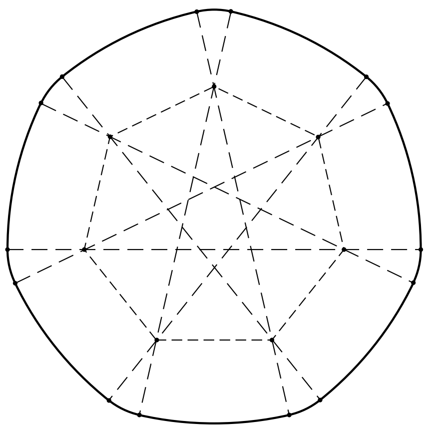

Fig. 8

Tracing out a locus

Any curve of constant width can turn inside a square; throughout the motion, the curve will always be in contact with all four sides of the square. This is illustrated for various positions of the Reuleaux triangle in Fig. 9.

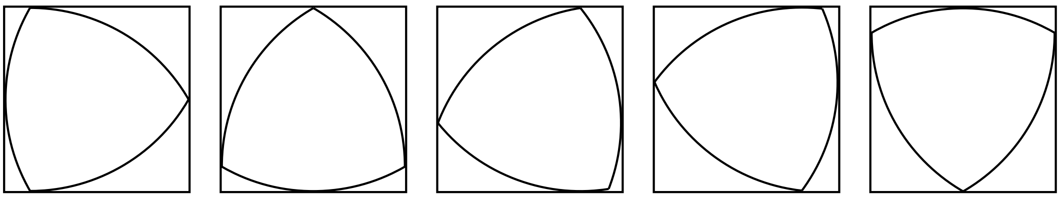

Fig. 9

If you trace the path followed by a vertex of the Reuleaux triangle, you will find that the locus is close to a square. This is shown in Fig. 10a; the locus of vertex P is made up of four straight line segments joined by rounded corners.

Fig. 10b shows the locus of the midpoint, Q, of one side of the equilateral triangle; this, too, is close to a square.

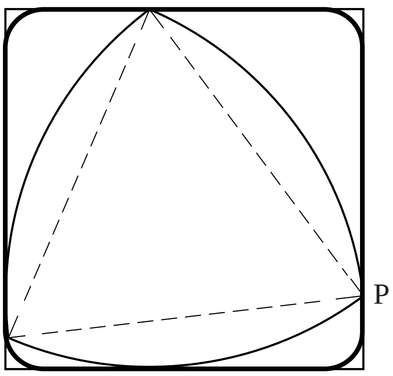

Fig. 10a

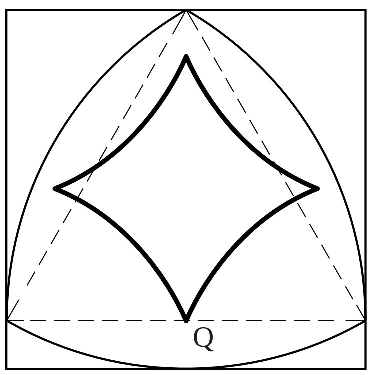

Fig. 10b

Other points in the Reuleaux triangle trace out other paths but none of these is a perfect square. This suggests the following question.

Is it possible to design a shape of constant width that contains a point which traces out a perfect square as the shape turns inside a square?

The answer to this question is ‘Yes’ and the curve is described below.

Drilling a square hole

Figs. 11a and 11b show how to construct a particular curve of constant width based on an isosceles triangle.

In triangle ABC, AB = *l* and angle CAB = angle CBA = 45°. Sides AC and BC are extended so that AE = BD = *l*. Arc BE has centre A, arc DA has centre B and arcs AB and ED have centre C.

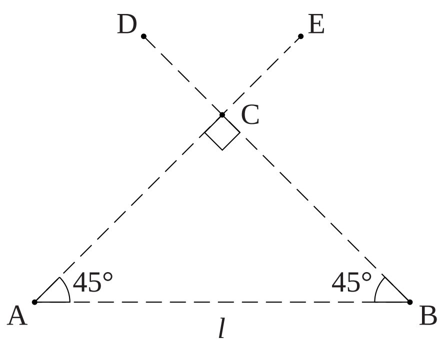

Fig. 11a

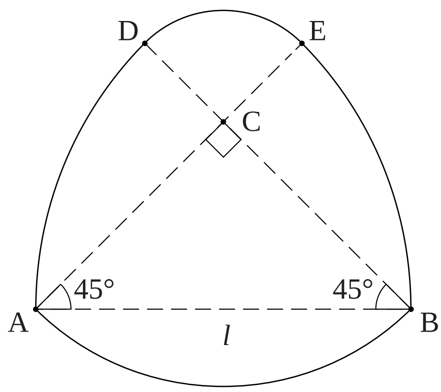

Fig. 11b

Like any curve of constant width *l*, the curve shown in Fig. 11b can turn inside a square of side length *l*. Fig. 12 shows this curve in various positions as it turns inside a square.

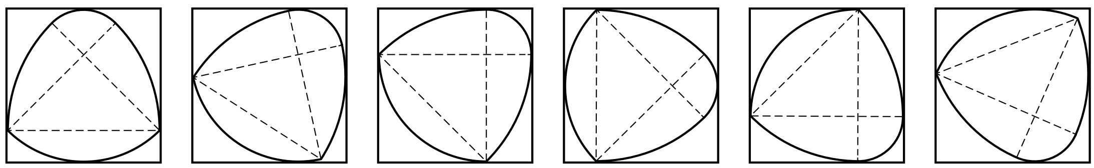

Fig. 12

The arc labelled DE in Fig. 11b remains in contact with the square throughout this motion. It follows that the locus of C is a perfect square.

This shape is used to drill a square hole.

A drill bit, with cross-section as shown in Fig. 11b, has a cutting tool at C. A metal guide plate, in which a square hole of side length *l* has been cut, is placed parallel to the material to be drilled. As the drill bit turns inside the guide plate, the cutting tool cuts out a square hole.

Other applications

Awareness of the existence of curves of constant width is important in engineering. Engineers rely on precision tools, many of which must be circular. To test that an object is circular, it is not sufficient simply to check that its width is constant; that would only imply that the object was one of the many shapes of constant width. Other means of checking for circularity, such as using circular templates, are needed.

It is possible that you are currently in possession of several shapes of constant width. Both the 50 pence and 20 pence coins have constant width (see Fig. 13). These coins were designed in this way so that they can easily be identified when used in machines; their width can be measured in any orientation as they move through the machine.

Fig. 13