C4 Specimen (MA)

Q1) $(4-3x)^{\frac{1}{2}} = 4^{-\frac{1}{2}}\left(1-\frac{3}{4}x\right)^{\frac{-1}{2}} = \frac{1}{2}\left(1-\frac{3}{4}x\right)^{-\frac{1}{2}}$

$$\frac{1}{2}\left(1-\frac{3x}{4}\right)^{-\frac{1}{2}} \simeq \frac{1}{2}\left[1+\frac{3x}{8}+\frac{-\frac{1}{2}\left(-\frac{3}{2}\right)}{2}\left(\frac{-3x}{4}\right)^2+\frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{6}\left(\frac{-3x}{4}\right)^3\right]$$

$$\left[\begin{array}{l} x=\frac{-3x}{4} \\ n=-\frac{1}{2} \end{array}\right]$$

$$\simeq \frac{1}{2}\left[1+\frac{3x}{8}+\frac{27}{128}x^2+\frac{135}{1024}x^3\right]$$

$$\simeq \boxed{\frac{1}{2}+\frac{3x}{16}+\frac{27}{256}x^2+\frac{135}{2048}x^3}$$

Q2) $\frac{d}{dx}\left(13x^2+13y^2-10xy=52\right)$

$$26x+26y\frac{dy}{dx}-10y-10x\frac{dy}{dx}=0.$$

$$\frac{dy}{dx}(26y-10x) = 10y-26x$$

$$\frac{dy}{dx} = \frac{10y-26x}{26y-10x} = \boxed{\frac{5y-13x}{13y-5x}}$$

Q3) $\displaystyle\int_0^1 \frac{1}{(1+x^2)^2}\,dx$

$x=\tan\theta$

$\frac{dx}{d\theta}=\sec^2\theta$.

$dx=\sec^2\theta\,d\theta$.

$\downarrow$

$$\int_0^{\frac{\pi}{4}}\left[\frac{1}{(1+\tan^2\theta)^2}\times\sec^2\theta\right]d\theta$$

//

x	θ	
0	0	(arctan 0)
1	$\frac{\pi}{4}$	(arctan 1)

$1+\tan^2\theta = \sec^2\theta$

$$= \int_0^{\frac{\pi}{4}} \left[\frac{1}{(\sec^2\theta)^2} \times \sec^2\theta\right] d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{\sec^2\theta}\, d\theta = \int_0^{\frac{\pi}{4}} \cos^2\theta\, d\theta \; //$$

$$\cos 2\theta = 2\cos^2\theta - 1$$

$$\frac{\cos 2\theta + 1}{2} = \cos^2\theta \; //$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} (\cos 2\theta + 1)\, d\theta = \frac{1}{2}\left[\frac{1}{2}\sin 2\theta + \theta\right]_0^{\pi/4}$$

$$= \frac{1}{2}\left[\frac{1}{2}\sin\frac{\pi}{2} + \frac{\pi}{4}\right] - \frac{1}{2}[0]$$

$$= \frac{1}{2}\left[\frac{1}{2} + \frac{\pi}{4}\right] = \boxed{\frac{1}{4} + \frac{\pi}{8}}$$

$\square$

Q4a) $x = \tan t \longrightarrow \frac{dx}{dt} = \sec^2 t$

$y = \sin 2t \longrightarrow \frac{dy}{dt} = 2\cos 2t$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2\cos 2t}{\sec^2 t} //$$

$\underline{t = \frac{\pi}{3}}$: $\frac{dy}{dx} = \frac{2\cos\left(\frac{2\pi}{3}\right)}{\frac{1}{\left(\cos\frac{\pi}{3}\right)^2}} = \boxed{-\frac{1}{4}}$

b) normal at P will have gradient 4. $\left(-\frac{1}{4} \times 4 = -1\right)$

$x = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$
$y = \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$\Rightarrow y - \frac{\sqrt{3}}{2} = 4\left(x - \sqrt{3}\right)$

$\Rightarrow y = 4x - 4\sqrt{3} + \frac{\sqrt{3}}{2}$

$\Rightarrow y = 4x - \frac{7}{2}\sqrt{3}$

c) $\underline{t = \frac{\pi}{4}}$: $\left.\begin{aligned} x &= \tan\frac{\pi}{4} = 1 \\ y &= \sin\frac{\pi}{2} = 1 \end{aligned}\right\}$ $\frac{dy}{dx} = \frac{2\cos\frac{\pi}{2}}{\frac{1}{\cos\left(\frac{\pi}{4}\right)^2}} = 0 //$

so normal at Q is the normal to the turning point //

hence $\boxed{x = 1}$

Q5a) $\begin{pmatrix} 5+\lambda \\ 3-2\lambda \\ -2+2\lambda \end{pmatrix} = \begin{pmatrix} 2-3\mu \\ -11-4\mu \\ a+5\mu \end{pmatrix}$ ~ ① ~ ②

①: $\lambda = -3-3\mu$

→ ②: $3-2(-3-3\mu) = -11-4\mu$

$3+6+6\mu+4\mu+11=0$

$10\mu+20=0$

$\mu=-2$ // $\rightarrow \lambda = -3-3(-2) = 3$ //

so point of intersection: $\begin{pmatrix} 2-3(-2) \\ -11-4(-2) \\ a+5(-2) \end{pmatrix} = \boxed{\begin{pmatrix} 8 \\ -3 \\ a-10 \end{pmatrix}}$ //

in terms of a.

OR: $\begin{pmatrix} 5+3 \\ 3-2(3) \\ -2+2(3) \end{pmatrix} = \boxed{\begin{pmatrix} 8 \\ -3 \\ 4 \end{pmatrix}}$

b) from (a), $4 = a-10$

$\boxed{a=14}$

c) using direction vectors: $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -4 \\ 5 \end{pmatrix} = -3+8+10$

$= 15$ //

$$\left|\begin{pmatrix}1\\-2\\2\end{pmatrix}\right| = \sqrt{1^2+2^2+2^2} = 3$$

$$\left|\begin{pmatrix}-3\\-4\\5\end{pmatrix}\right| = \sqrt{3^2+4^2+5^2} = 5\sqrt{2}$$

$$\therefore \cos\theta = \frac{15}{15\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\therefore \theta = \cos^{-1}\frac{1}{\sqrt{2}} = \boxed{45^\circ}$$

Q6a) $11x-1 = A(2+3x) + B(1-x)(2+3x) + C(1-x)^2$

$\underline{x=1}$: $10 = 5A$ $\therefore \boxed{A = 2}$

$2+3x=0$
$x = -\frac{2}{3}$

$\underline{x = -\frac{2}{3}}$: $-\frac{25}{3} = \frac{25}{9}C$ $\therefore \boxed{C = -3}$

$\underline{x=0}$: $-1 = 2A + 2B + C$

$-1 = 4 - 3 + 2B$

$\therefore B = \frac{-1+3-4}{2} = \boxed{-1 = B}$

b) $$\int_0^{\frac{1}{2}} \left[\frac{2}{(1-x)^2} - \frac{1}{(1-x)} - \frac{3}{(2+3x)}\right] dx$$

$$= 2\int_0^{\frac{1}{2}} \left[\frac{1}{(1-x)^2}\right] dx - \int_0^{\frac{1}{2}} \left(\frac{1}{1-x}\right) dx - 3\int_0^{\frac{1}{2}} \left(\frac{1}{2+3x}\right) dx$$

$$= 2\left[\frac{\cancel{-}(1-x)^{-1}}{\cancel{-1}}\right]_0^{\frac{1}{2}} - \left[-\ln|1-x)\right]_0^{\frac{1}{2}}$$

$$- \left[\frac{3}{3}\ln|2+3x|\right]_0^{\frac{1}{2}}$$

$$= 2\left[\frac{1}{\frac{1}{2}}\right] - 2\left[1\right] - \left[-\ln\frac{1}{2}\right] + \left[0\right]$$

$$- \left[\ln\frac{7}{2}\right] + \left[\ln 2\right]$$

$$= 4 - 2 + \ln\frac{1}{2} - \ln\frac{7}{2} + \ln 2$$

$$= \boxed{2 + \ln\frac{2}{7}}$$

Q7a) $u = \frac{x}{2} - \frac{1}{8}\sin 4x$

$$\frac{du}{dx} = \frac{1}{2} - \frac{1}{2}\cos 4x = \frac{1}{2}(1-\cos 4x)$$

$$= \frac{1}{2}(1-(1-2\sin^2 2x))$$

$$= \frac{1}{2}(1-1+2\sin^2 2x) = \sin^2 2x$$

b) $V = \pi\int_0^{\pi/4} y^2\,dx = \pi\int_0^{\pi/4}(x\sin^2 2x)\,dx$

<u>sub</u>: $u = \frac{x}{2} - \frac{1}{8}\sin 4x$

$$\frac{du}{dx} = \sin^2 2x \rightarrow dx = \frac{du}{\sin^2 2x}$$

x	u
0	0
$\frac{\pi}{4}$	$\frac{\pi}{8}$

$x=0$: $u=0$

$x=\frac{\pi}{4}$: $u = \frac{\pi}{8} - \frac{1}{8}\sin\pi = \frac{\pi}{8}$

$$\Rightarrow V = \pi\int_0^{\pi/8}\left[x\sin^2 2x\right]\times\frac{1}{\sin^2 2x}\,du$$

$$V = \pi\int_0^{\pi/8}[x]\,du = \pi\int_0^{\pi/8}\left[2u+\frac{1}{4}\sin 4x\right]du$$

$$= \frac{\pi^3}{64} + \frac{\pi}{16} = \boxed{\frac{\pi}{64}\left(\pi^2+4\right)}$$

Q8a) $\frac{dr}{dt} \propto \frac{1}{r^2}$

rate of increase of radius

inverse of the square of radius.

$$\Rightarrow \frac{dr}{dt} = \frac{k}{r^2}.$$

$$\boxed{\frac{dA}{dt} = \frac{dr}{dt} \times \frac{dA}{dr}}$$

$A = \pi r^2$

$\therefore \frac{dA}{dr} = 2\pi r$ //

$$\Rightarrow \frac{dA}{dt} = \frac{k}{r^2} \times 2\pi r = \frac{2k\pi}{r}$$

$$\Rightarrow \frac{dA}{dt} = 2k\pi\left(\frac{1}{r}\right) = 2k\pi\left(\frac{1}{\sqrt{\frac{A}{\pi}}}\right) = \frac{2k\pi^{\frac{3}{2}}}{\sqrt{A}}$$ //

$A = \pi r^2$

$\therefore \sqrt{\frac{A}{\pi}} = r$

hence $\frac{dA}{dt} \propto \frac{1}{\sqrt{A}}$ ($2k\pi^{3/2}$ is a constant)

b) $\frac{dS}{dt} = \frac{2e^{2t}}{\sqrt{S}}$

$\sqrt{S}\,\frac{dS}{dt} = 2e^{2t}$

$\int S^{\frac{1}{2}}\,dS = 2\int e^{2t}\,dt$

$\frac{2}{3}S^{\frac{3}{2}} = e^{2t} + c$

$\underline{t=0,\ S=9}$: $\frac{2}{3}(9)^{1.5} = 1 + c$

$c = 18 - 1 = 17$ //

so $\frac{2}{3}S^{\frac{3}{2}} = e^{2t} + 17$

$\underline{S=16}$: $\frac{2}{3}(16)^{1.5} - 17 = e^{2t} = \frac{77}{3}$

$2t = \ln\frac{77}{3}$

$\therefore t = \frac{1}{2}\ln\frac{77}{3} = \boxed{1.6}$