C4 June 2018 (MA)

Q1a) $\sqrt{4-9x} = (4-9x)^{\frac{1}{2}} = 4^{\frac{1}{2}}\left(1-\frac{9}{4}x\right)^{\frac{1}{2}}$

$= 2\left(1-\frac{9}{4}x\right)^{\frac{1}{2}} \simeq 2\left[1-\frac{9}{8}x + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2}\left(-\frac{9}{4}x\right)^2\right]$

$\left[\begin{array}{l} n=\frac{1}{2} \\ x=-\frac{9}{4}x \end{array}\right]$ $\simeq 2\left[1-\frac{9}{8}x - \frac{81}{128}x^2\right]$

$\simeq \boxed{2 - \frac{9}{4}x - \frac{81}{64}x^2}$

b) note that $|x| < \frac{4}{9}$.

so $4-9x \neq 310$

instead try: $4-9x = 3.1$

$9x = 0.9$

$\therefore x = 0.1$

$\therefore \sqrt{3.1} \simeq 2 - \frac{9}{4}(0.1) - \frac{81}{64}(0.1)^2$

$\sqrt{3.1} \simeq 1.7623..$

$\sqrt{310} = 10 \times \sqrt{3.1} \simeq \boxed{17.623}$

Q2a) $\frac{d}{dx}\left(x^2+xy+y^2-4x-5y+1=0\right)$

$$2x+y+x\frac{dy}{dx}+2y\frac{dy}{dx}-4-5\frac{dy}{dx}=0$$

$$\frac{dy}{dx}(x+2y-5)=4-y-2x$$

$$\therefore \frac{dy}{dx}=\boxed{\frac{4-y-2x}{x+2y-5}}$$

b) $$\frac{4-y-2x}{x+2y-5}=0$$

$$\therefore 4-y-2x=0$$

$$2x+y=4$$

$$\hookrightarrow y=4-2x$$

<u>sub y into C</u> : $x^2+x(4-2x)+(4-2x)^2-4x-5(4-2x)+1=0$

$\Rightarrow x^2+\cancel{4x}-2x^2+16-16x+4x^2-\cancel{4x}-20+10x+1=0$

$\Rightarrow 3x^2-6x-3=0$

$\Rightarrow x^2-2x-1=0$

<u>By Quadratic Formula</u> : $\boxed{x=1+\sqrt{2}}$ $\boxed{x=1-\sqrt{2}}$

Q3ai) $\dfrac{13-4x}{(2x+1)^2(x+3)} = \dfrac{A(2x+1)(x+3)+B(x+3)+C(2x+1)^2}{(2x+1)^2(x+3)}$

$\therefore 13-4x = A(2x+1)(x+3) + B(x+3) + C(2x+1)^2$

$\underline{x=-3}$: $25 = 25C$ $\therefore \boxed{C=1}$

$\underline{x=-\frac{1}{2}}$: $13+2 = \frac{5}{2}B$ $\therefore \boxed{B=6}$

$\underline{x=0}$: $13 = 3A+18+1$ $\therefore \boxed{A=-2}$

b) $\displaystyle\int\left[\frac{-2}{2x+1}+\frac{6}{(2x+1)^2}+\frac{1}{x+3}\right]dx$

$= -\ln|2x+1| + 6\left[\dfrac{(2x+1)^{-1}}{-1\times 2}\right] + \ln|x+3|$ $(+c)$

$$\boxed{= -\ln|2x+1| - 3(2x+1)^{-1} + \ln|x+3| + c}$$

ii) $\int (e^x+1)^3 \, dx$

$= \int (e^{2x}+2e^x+1)(e^x+1) \, dx$

$= \int (e^{3x}+e^{2x}+2e^{2x}+2e^x+e^x+1) \, dx$

$= \int \left[e^{3x}+3e^{2x}+3e^x+1\right] dx$

$= \boxed{\frac{1}{3}e^{3x}+\frac{3}{2}e^{2x}+3e^x+x+c}$

iii) $\int \left[\frac{1}{4x+5x^{\frac{1}{3}}}\right] dx$

$\left[\begin{array}{l} u^3 = x \\ u = x^{\frac{1}{3}} \\ 3u^2 \frac{du}{dx} = 1 \quad \therefore \; dx = 3u^2 \, du \end{array}\right]$

$\Rightarrow \int \left[\frac{3u^2}{4u^3+5u}\right] du = \int \left[\frac{3u}{4u^2+5}\right] du$

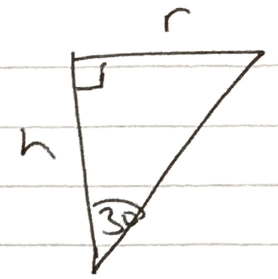
By Pattern!

$= \frac{3}{8} \ln|4u^2+5| + c$

$u^2 = x^{\frac{2}{3}}$.

$= \boxed{\frac{3}{8} \ln|4x^{\frac{2}{3}}+5| + c}$

Q4a) $V = \frac{1}{3}\pi r^2 h$



$$\tan 30 = \frac{r}{h} = \frac{\sqrt{3}}{3}$$

$$\therefore \quad r = \frac{\sqrt{3}}{3}h$$

so $V = \frac{1}{3}\pi\left(\frac{\sqrt{3}}{3}h\right)^2 \times h$

$$= \frac{1}{3}\pi \times \frac{1}{3}h^2 \times h = \frac{1}{9}\pi h^3$$

b) $\frac{dV}{dt} = 200.$

we want $\frac{dh}{dt}$ . . .

$$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$$

$$V = \frac{1}{9}\pi h^3$$

$$\therefore \frac{dV}{dh} = \frac{1}{3}\pi h^2 \qquad \therefore \frac{dh}{dV} = \frac{3}{\pi h^2}$$

$$\therefore \frac{dh}{dt} = \frac{200 \times 3}{\pi h^2} = \frac{600}{\pi h^2}.$$

$$\underline{h=15}: \quad \frac{dh}{dt} = \frac{600}{\pi(15)^2} = \boxed{\frac{8}{3\pi}} \text{ cms}^{-1}$$

Q5a) $\underline{y=2}$: $\quad 2 = 2 - 4\cos t$

$$\cos t = 0$$

$$t = \frac{\pi}{2}, \ -\frac{\pi}{2},$$

$$x = 1 + \frac{\pi}{2} - 5 = -2.43.. < 0$$

so reject

$$x = 1 - \frac{\pi}{2} - 5\sin\left(-\frac{\pi}{2}\right)$$
$$= 7.57.. > 0 \ //$$

$$k > 0 \text{ so } k = 1 - \frac{\pi}{2} + 5\sin\frac{\pi}{2}$$

$$k = 1 + 5 - \frac{\pi}{2} = \boxed{6 - \frac{\pi}{2}}$$

$$\left(= \frac{12-\pi}{2}\right)$$

b) $x = 1 + t - 5\sin t \qquad y = 2 - 4\cos t$

$$\frac{dx}{dt} = 1 - 5\cos t \qquad \frac{dy}{dt} = 4\sin t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4\sin t}{1 - 5\cos t} = \frac{4\sin\left(\frac{-\pi}{2}\right)}{1 - 5\cos\left(\frac{-\pi}{2}\right)}$$

$= -4 = m_{tangent}$.

$\therefore y - 2 = -4(x - (6 - \frac{\pi}{2}))$

$y = -4x + 4(6 - \frac{\pi}{2}) + 2$

$y = -4x + 24 - 2\pi + 2$

$\boxed{y = -4x + 26 - 2\pi}$

Q6) $\frac{dy}{dx} = \frac{y^2}{3\cos^2 2x}$

$\left(\frac{1}{y^2}\right)\frac{dy}{dx} = \left(\frac{1}{3\cos^2 2x}\right)$

$\int (y^{-2})\, dy = \frac{1}{3}\int (\sec^2 2x)\, dx$

$\frac{y^{-1}}{-1} = \frac{1}{3}\left[\frac{1}{2}\tan 2x\right] + c$

$-\frac{1}{y} = \frac{1}{6}\tan 2x + c$

$-y = \frac{1}{\frac{1}{6}\tan 2x + c}$

$y = \frac{-1}{\frac{1}{6}\tan 2x + c}$

at $y = 2$, $x = \frac{-\pi}{8}$ $\therefore$ $2 = \dfrac{-1}{\frac{1}{6}\tan\frac{-\pi}{4} + c}$

$\therefore$ $2 = \dfrac{-1}{\frac{-1}{6} + c}$

$\Rightarrow -\frac{1}{3} + 2c = -1$

$\Rightarrow 2c = -\frac{2}{3}$ $\therefore$ $c = -\frac{1}{3}$ //

so ... $\boxed{y = \dfrac{-1}{\frac{1}{6}\tan 2x - \frac{1}{3}}}$ o.e

Q7a) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$\therefore \overrightarrow{OB} = \overrightarrow{AB} + \overrightarrow{OA}$

$= \begin{pmatrix} 4 \\ -6 \\ 2 \end{pmatrix} + \begin{pmatrix} -3 \\ 7 \\ 2 \end{pmatrix} = \boxed{\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}}$

b) $P\begin{pmatrix} 9 \\ 1 \\ 8 \end{pmatrix}$ $\therefore \overrightarrow{PA} = \overrightarrow{OA} - \overrightarrow{OP} = \begin{pmatrix} -3 \\ 7 \\ 2 \end{pmatrix} - \begin{pmatrix} 9 \\ 1 \\ 8 \end{pmatrix}$

$= \begin{pmatrix} -12 \\ 6 \\ -6 \end{pmatrix}$ //

and $\overrightarrow{BA} = \begin{pmatrix} -4 \\ 6 \\ -2 \end{pmatrix}$ //

[equivalently you can use $\overrightarrow{AP}$ and $\overrightarrow{AB}$]

$$\overrightarrow{PA} \cdot \overrightarrow{BA} = \begin{pmatrix} -12 \\ 6 \\ -6 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ 6 \\ -2 \end{pmatrix} = 96 //$$

$$|\overrightarrow{PA}| = \sqrt{12^2 + 6^2 + 6^2} = 6\sqrt{6} //$$

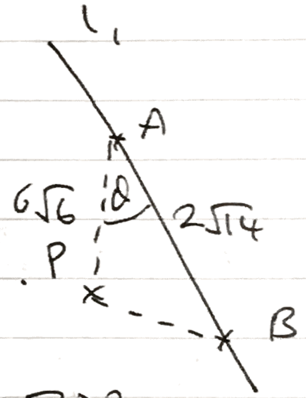
$$|\overrightarrow{BA}| = \sqrt{4^2 + 6^2 + 2^2} = 2\sqrt{14} //$$

$$\therefore \cos\theta = \frac{96}{6\sqrt{6} \times 2\sqrt{14}} = \boxed{\frac{4\sqrt{21}}{21}}$$

unneeded. $\left\{ \therefore \theta = \cos^{-1}\left(\frac{4\sqrt{21}}{21}\right) = 29.2^\circ \right.$

c) Area $= \frac{1}{2} ab\sin C$

$$= \frac{1}{2} \times 6\sqrt{6} \times 2\sqrt{14} \times \sin\theta$$



$$\cos\theta = \frac{4\sqrt{21}}{21}$$

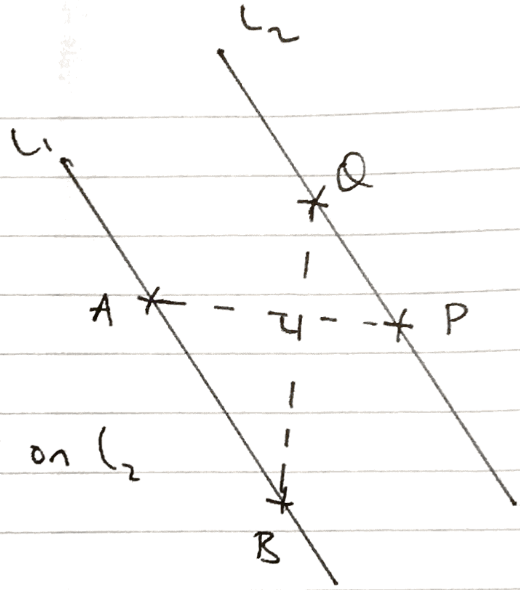
$$\sin^2\theta = 1 - \cos^2\theta = 1 - \left(\frac{4\sqrt{21}}{21}\right)^2 = \frac{5}{21} //$$

$$\therefore \sin\theta = \sqrt{\frac{5}{21}}$$

so Area $= \frac{1}{2} \times 6\sqrt{6} \times 2\sqrt{14} \times \sqrt{\frac{5}{21}}$

$$= \boxed{12\sqrt{5}}$$

d) l_2 : $\boxed{r = \begin{pmatrix} 9 \\ 1 \\ 8 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -6 \\ 2 \end{pmatrix}}$



e) $\overrightarrow{AP} = \begin{pmatrix} 12 \\ -6 \\ 6 \end{pmatrix}$

$\overrightarrow{BQ} = \overrightarrow{OQ} - \overrightarrow{OB}$

$\overrightarrow{OQ} = \begin{pmatrix} 9+4\lambda \\ 1-6\lambda \\ 8+2\lambda \end{pmatrix}$ as Q is on l_2

$\therefore \overrightarrow{BQ} = \begin{pmatrix} 9+4\lambda \\ 1-6\lambda \\ 8+2\lambda \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 8+4\lambda \\ -6\lambda \\ 4+2\lambda \end{pmatrix}$ //

now recall that AP is perpendicular to BQ.

$\therefore \overrightarrow{AP} \cdot \overrightarrow{BQ} = 0$

so $\begin{pmatrix} 12 \\ -6 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 8+4\lambda \\ -6\lambda \\ 4+2\lambda \end{pmatrix} = 0$

$\Rightarrow 96 + 48\lambda + 36\lambda + 24 + 12\lambda = 0$

$\Rightarrow 120 + 96\lambda = 0$

$\Rightarrow \lambda = \frac{-5}{4}$ //

$\therefore \overrightarrow{OQ} = \begin{pmatrix} 9+4(\frac{-5}{4}) \\ 1-6(\frac{-5}{4}) \\ 8+2(\frac{-5}{4}) \end{pmatrix} = \boxed{\begin{pmatrix} 4 \\ 17/2 \\ 11/2 \end{pmatrix}}$

Q8a) $\int (x\cos 4x)\,dx$ ~ <u>By Parts</u>

$\frac{dv}{dx} = \cos 4x$ $\quad u = x$

$v = \frac{1}{4}\sin 4x$ $\quad u' = 1$

$= \frac{1}{4}x\sin 4x - \frac{1}{4}\int [\sin 4x]\,dx$

$= \frac{1}{4}x\sin 4x - \frac{1}{4}\left[-\frac{1}{4}\cos 4x\right] + c$

$= \boxed{\frac{1}{4}x\sin 4x + \frac{1}{16}\cos 4x + c}$

b) $V.R = \pi \int_0^{\frac{\pi}{4}} [y^2]\,dx = \pi \int_0^{\pi/4} [x\sin^2 2x]\,dx$

$\cos 4x = 1 - 2\sin^2 2x$

$\frac{1-\cos 4x}{2} = \sin^2 2x$

$\therefore R = \frac{\pi}{2}\int_0^{\pi/4} [x(1-\cos 4x)]\,dx$

$$= \frac{\pi}{2}\int_0^{\pi/4} [x - x\cos 4x]\, dx$$

$$= \frac{\pi}{2}\int_0^{\pi/4} [x]\, dx - \frac{\pi}{2}\int_0^{\pi/4} [x\cos 4x]\, dx$$

$$= \frac{\pi}{2}\left[\frac{x^2}{2}\right]_0^{\pi/4} - \frac{\pi}{2}\left[\frac{1}{4}x\sin 4x + \frac{1}{16}\cos 4x\right]_0^{\pi/4}$$

$$= \frac{\pi}{2}\left[\frac{\pi^2}{32}\right] - \frac{\pi}{2}\left[-\frac{1}{16}\right] + \frac{\pi}{2}\left[\frac{1}{16}\right]$$

$$= \pi\left(\frac{\pi^2}{64} + \frac{1}{16}\right) = \boxed{\frac{\pi}{16}\left(\frac{\pi^2}{4} + 1\right)}$$