

C4 June 2017 (MA)

Q1a)

$$x = 3t - 4$$

$$\frac{dx}{dt} = 3$$

$$y = 5 - 6t^{-1}$$

$$\frac{dy}{dt} = 6t^{-2}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6t^{-2}}{3} = \boxed{\frac{2}{t^2}}$$

b) at $t = \frac{1}{2}$: $x = \frac{3}{2} - 4 = -\frac{5}{2}$ } $P(-\frac{5}{2}, -7)$
 $y = 5 - \frac{6}{\frac{1}{2}} = -7$ }

at P, $\frac{dy}{dx} = \frac{2}{(\frac{1}{2})^2} = 8 \therefore m_{\text{tangent}} = 8.$

$$\Rightarrow y - (-7) = 8(x - (-\frac{5}{2}))$$

$$\Rightarrow y = 8x + 20 - 7$$

$$\Rightarrow \boxed{y = 8x + 13}$$

c) $y = 5 - \frac{6}{t}$

$$x = 3t - 4$$

$$\therefore \left(\frac{x+4}{3} = t\right) \xrightarrow{\text{sub into } y} y = 5 - \frac{6}{\frac{x+4}{3}}$$

$$\therefore y = 5 - \frac{18}{x+4}$$

$$y = \frac{5(x+4)}{x+4} - \frac{18}{x+4} = \frac{5x+20-18}{x+4}$$

$$\boxed{y = \frac{5x+2}{x+4}} \quad a=5$$

$$b=2$$

$$Q2a) (2+kx)^{-3} = 2^{-3} \left(1 + \frac{kx}{2}\right)^{-3} = \frac{1}{8} \left(1 + \frac{kx}{2}\right)^{-3}$$

$$\therefore \boxed{A = \frac{1}{8}}$$

$$b) \frac{1}{8} \left(1 + \frac{kx}{2}\right)^{-3} = \frac{1}{8} \left[1 - \frac{3kx}{2} + \frac{-3(-4)}{2} \left(\frac{kx}{2}\right)^2 \right]$$

$$\left[\begin{array}{l} n = -3 \\ x = \frac{kx}{2} \end{array} \right] = \frac{1}{8} \left[1 - \frac{3kx}{2} + \frac{3k^2x^2}{2} \right]$$

$$\approx \frac{1}{8} - \frac{3kx}{16} + \frac{3k^2x^2}{16} //$$

$$B = -\frac{3k}{16} \quad \text{and} \quad \frac{3k^2}{16} = \frac{243}{16}$$

$$\Rightarrow 3k^2 = 243$$

$$\Rightarrow k^2 = \frac{243}{3} = 81$$

$$\Rightarrow k = \sqrt{81} = \boxed{9} \quad (k > 0)$$

$$c) \therefore B = -\frac{3}{16}(9) = \boxed{\frac{-27}{16}}$$

Q3a)

x	$0-2$
y	1.86254

$$b) h = \frac{b-a}{n} = \frac{1-0}{5} // = \frac{1}{5}$$

$$\text{Area} \approx \frac{1}{2} \times \frac{1}{5} \left[2 + 1.27165 + 2(1.41994 + 1.56981 + 1.7183) + 1.86254 \right]$$

$$\approx \boxed{1.6413}$$

$$c) \text{Area} = \int_0^1 \left(\frac{6}{e^x + 2} \right) dx$$

$$\left[\begin{array}{l} u = e^x \\ \frac{du}{dx} = e^x \\ \therefore dx = e^{-x} du \end{array} \right]$$

$$= \int_1^e \left[\frac{6}{u+2} \times e^{-x} \right] du$$

$$\left[\begin{array}{c|c} x & u \\ \hline 0 & 1 \\ 1 & e \end{array} \right]$$

$$= \int_1^e \left[\frac{6}{u+2} \times \frac{1}{u} \right] du$$

$$(e^{-x} = \frac{1}{u})$$

$$= \int_1^e \left[\frac{6}{u(u+2)} \right] du // \square$$

$$d) R = 6 \int_1^e \left[\frac{1}{u(u+2)} \right] du \Rightarrow \text{By partial fractions...}$$

$$\frac{1}{u(u+2)} = \frac{A}{u} + \frac{B}{u+2}$$

$$1 = A(u+2) + B(u)$$

$$\underline{u=0} : 1 = 2A \quad \therefore A = \frac{1}{2} //$$

$$\underline{u=1} : 1 = 3A + B \quad \therefore B = 1 - \frac{3}{2} = -\frac{1}{2} //$$

$$R = 6 \int_1^e \left[\frac{\frac{1}{2}}{u} - \frac{\frac{1}{2}}{u+2} \right] du$$

$$= 6 \left[\frac{1}{2} \ln|u| - \frac{1}{2} \ln|u+2| \right]_1^e$$

$$= 6 \left[\frac{1}{2} \ln \left| \frac{u}{u+2} \right| \right]_1^e = 6 \left[\frac{1}{2} \ln \frac{e}{e+2} \right] - 6 \left[\frac{1}{2} \ln \frac{1}{3} \right]$$

$$= 3 \ln \frac{e}{e+2} - 3 \ln \frac{1}{3} = \boxed{3 \ln \left(\frac{3e}{e+2} \right) //}$$

or equivalent.
(many forms)

$$\bullet \text{ Q4a) } \frac{d}{dx} (4x^2 - y^3 - 4xy + 2^y = 0)$$

$$\Rightarrow 8x - 3y^2 \frac{dy}{dx} - 4y - 4x \frac{dy}{dx} + (2^y \ln 2) \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} (2^y \ln 2 - 4x - 3y^2) = 4y - 8x$$

$$\Rightarrow \frac{dy}{dx} = \frac{4y - 8x}{2^y \ln 2 - 4x - 3y^2} //$$

$$\text{at P, } \frac{dy}{dx} = \frac{4(4) - 8(-2)}{2^4 \ln 2 - 4(-2) - 3(16)} = \boxed{\frac{32}{16 \ln 2 - 40}}$$

$$\text{b) at normal to P, } m = \frac{-(16 \ln 2 - 40)}{32} = \frac{40 - 16 \ln 2}{32}$$

$$\Rightarrow y - 4 = \frac{40 - 16 \ln 2}{32} (x + 2)$$

$$\underline{x=0}: y = 4 + \frac{2(40 - 16 \ln 2)}{32}$$

$$y = 4 + \frac{40 - 16 \ln 2}{16}$$

$$y = \frac{64 + 40 - 16 \ln 2}{16} = \boxed{\frac{13}{2} - \ln 2}$$

$$\begin{aligned}
 \text{Q5) } V &= \pi \int_0^{\ln 4} (y^2) dx = \pi \int_0^{\ln 4} [(e^x + 2e^{-x})(e^x + 2e^{-x})] dx \\
 &= \pi \int_0^{\ln 4} [e^{2x} + 4 + 4e^{-2x}] dx \\
 &= \pi \left[\frac{1}{2} e^{2x} + 4x - 2e^{-2x} \right]_0^{\ln 4} \\
 &= \pi \left[\frac{1}{2} e^{2\ln 4} + 4\ln 4 - 2e^{-2\ln 4} \right] - \pi \left[\frac{1}{2} - 2 \right] \\
 &= \pi \left[\frac{1}{2} e^{\ln 16} + 4\ln 4 - 2e^{\ln \frac{1}{16}} \right] + \frac{3\pi}{2} \\
 &= \pi \left[8 + 4\ln 4 - \frac{2}{16} + \frac{3}{2} \right] = \pi \left[\frac{75}{8} + 4\ln 4 \right] \\
 &= \boxed{\pi \left[\frac{75}{8} + \ln 256 \right]}
 \end{aligned}$$

$$(Q6a) \quad \begin{pmatrix} 4 - \lambda \\ 28 - 5\lambda \\ 4 + \lambda \end{pmatrix} = \begin{pmatrix} 5 + 3\mu \\ 3 \\ 1 - 4\mu \end{pmatrix} \quad \begin{array}{l} \sim \textcircled{1} \\ \sim \textcircled{2} \\ \sim \textcircled{3} \end{array}$$

$$\textcircled{2} : \therefore 28 - 5\lambda = 3$$

$$25 = 5\lambda \quad \therefore \lambda = 5 //$$

$$\lambda = 5 : X = \begin{pmatrix} 4 - 5 \\ 28 - 25 \\ 4 + 5 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 9 \end{pmatrix}$$

$$b) \text{ using dir. vectors : } \begin{pmatrix} -1 \\ -5 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix} = -3 - 4 = -7 //$$

$$\left| \begin{pmatrix} -1 \\ -5 \\ 1 \end{pmatrix} \right| = \sqrt{1^2 + 5^2 + 1^2} = 3\sqrt{3}$$

$$\left| \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix} \right| = \sqrt{3^2 + 4^2} = 5$$

$$\cos \theta = \frac{-7}{5 \cdot 3\sqrt{3}}$$

$$= \frac{-7\sqrt{3}}{45} //$$

$$\therefore \theta = \cos^{-1}\left(\frac{-7\sqrt{3}}{45}\right)$$

$$= 105.63^\circ$$

$$\text{so required angle} = 180 - 105.63$$

$$= \boxed{74.37^\circ}$$

$$c) \vec{AX} = \vec{OX} - \vec{OA} = \begin{pmatrix} -1 \\ 3 \\ 9 \end{pmatrix} - \begin{pmatrix} 2 \\ 18 \\ 6 \end{pmatrix} = \begin{pmatrix} -3 \\ -15 \\ 3 \end{pmatrix} //$$

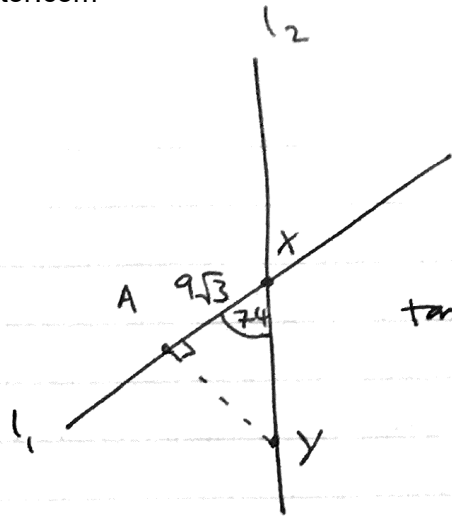
$$\therefore |\vec{AX}| = \sqrt{3^2 + 15^2 + 3^2} = \boxed{9\sqrt{3}}$$

$$d) |\vec{AX}| = 9\sqrt{3}$$

$$\tan(74.37^\circ) = \frac{AY}{9\sqrt{3}}$$

$$\therefore AY = 9\sqrt{3} \tan(74.37)$$

$$= \boxed{55.7}$$



$$\tan \theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$e) |\vec{AB}| = \frac{1}{2} |\vec{AX}|$$

$$\therefore \vec{OB} = \vec{OA} \pm \frac{1}{2} \vec{AX}$$

$$\left. \begin{array}{l} \vec{AX} = \begin{pmatrix} -3 \\ -15 \\ 3 \end{pmatrix} \\ \vec{OA} = \begin{pmatrix} 2 \\ 18 \\ 6 \end{pmatrix} \end{array} \right\} \vec{OB}_1 = \begin{pmatrix} 2 \\ 18 \\ 6 \end{pmatrix} + \begin{pmatrix} -3/2 \\ -15/2 \\ 3/2 \end{pmatrix} = \boxed{\begin{pmatrix} 1/2 \\ 21/2 \\ 15/2 \end{pmatrix}}$$

$$\text{and } \vec{OB}_2 = \begin{pmatrix} 2 \\ 18 \\ 6 \end{pmatrix} - \begin{pmatrix} -3/2 \\ -15/2 \\ 3/2 \end{pmatrix} = \boxed{\begin{pmatrix} 7/2 \\ 51/2 \\ 9/2 \end{pmatrix}}$$

$$Q7a) \quad \frac{dh}{dt} = k(h-9)^{\frac{1}{2}}$$

$$h=130, \quad \frac{dh}{dt} = -1.1 : \quad -1.1 = k(121)^{\frac{1}{2}}$$

$$\frac{-1.1}{11} = k = \boxed{-0.1}$$

$$b) \quad \frac{dh}{dt} = -0.1(h-9)^{\frac{1}{2}}$$

$$\left(\frac{1}{(h-9)^{\frac{1}{2}}} \right) \frac{dh}{dt} = -0.1$$

$$\int [(h-9)^{-\frac{1}{2}}] dh = -0.1 \int (1) dt$$

$$2(h-9)^{\frac{1}{2}} = -0.1t + c$$

$$\underline{h=200, t=0} : \quad 2(191)^{\frac{1}{2}} = c = 2\sqrt{191}$$

$$\therefore 2\sqrt{h-9} = 2\sqrt{191} - 0.1t$$

$$\underline{h=50} : \quad \frac{2\sqrt{41} - 2\sqrt{191}}{-0.1} = t = 148.343 \dots$$

$$= \boxed{148}$$

Q8a) $y=8$: $8 = \sec^3 \theta$
 $\frac{1}{8} = \cos^3 \theta \quad \therefore \cos \theta = \frac{1}{2}$
 $\theta = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

at $\theta = \frac{\pi}{3}$, $x = 3\left(\frac{\pi}{3}\right)\sin\frac{\pi}{3} = \boxed{\frac{\pi\sqrt{3}}{2}} = k$.

b) $R = \int \left(y \frac{dx}{d\theta}\right) d\theta \quad \left[\frac{dx}{d\theta} = 3\theta \cos \theta + 3 \sin \theta \right]$
 (PRODUCT RULE)

$= \int_0^{\pi/3} [\sec^3 \theta \cdot (3\theta \cos \theta + 3 \sin \theta)] d\theta$

x	θ
0	0
$\frac{\pi\sqrt{3}}{2}$	$\frac{\pi}{3}$

$\leftarrow y=8, \theta = \frac{\pi}{3}$

$= \int_0^{\pi/3} \left[3\theta \sec^2 \theta + \frac{3 \sin \theta}{\cos \theta} \cdot \sec^2 \theta \right] d\theta$

$= 3 \int_0^{\pi/3} [\theta \sec^2 \theta + \tan \theta \sec^2 \theta] d\theta$

$\lambda = 3$
 $\alpha = 0$
 $\beta = \frac{\pi}{3}$

$$c) R = 3 \int_0^{\pi/3} [\theta \sec^2 \theta] d\theta + 3 \int_0^{\pi/3} [\tan \theta \sec^2 \theta] d\theta$$

①
②

① By parts: $\frac{dv}{d\theta} = \sec^2 \theta \rightarrow v = \tan \theta$
 $u = \theta \rightarrow u' = 1$

$$= 3 \left[[\theta \tan \theta]_0^{\pi/3} - \int_0^{\pi/3} [\tan \theta] d\theta \right]$$

$$= 3 \left[\frac{\pi}{3} \sqrt{3} - [\ln |\sec \theta|]_0^{\pi/3} \right]$$

$$= \pi \sqrt{3} - 3 \ln 2 //$$

② By pattern: $3 \int_0^{\pi/3} [\tan \theta \sec \theta \cdot \sec \theta] d\theta$

$$= 3 \left[\frac{\sec^2 \theta}{2} \right]_0^{\pi/3} = 3 \left[2 - \frac{1}{2} \right] = \frac{9}{2} //$$

$$R = \textcircled{1} + \textcircled{2} = \boxed{\pi \sqrt{3} - 3 \ln 2 + \frac{9}{2}}$$