C4 - Withdrawn 2013 (MA)

Q1a) $\sqrt{9+8x}$

$$\Rightarrow (9+8x)^{\frac{1}{2}} = 9^{\frac{1}{2}}\left(1+\frac{8}{9}x\right)^{\frac{1}{2}} = 3\left(1+\frac{8}{9}x\right)^{\frac{1}{2}}$$

$$\Rightarrow 3\left(1+\frac{8}{9}x\right)^{\frac{1}{2}} \simeq 3\left[1+\frac{4}{9}x + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2}\left(\frac{8}{9}x\right)^2\right]$$

$$\begin{bmatrix} x = \frac{8}{9}x \\ n = \frac{1}{2} \end{bmatrix} \simeq \boxed{3 + \frac{4}{3}x - \frac{8}{27}x^2}$$

b) $9 + 8x = 11$

$8x = 2$

$x = \frac{1}{4}$

$$\therefore \sqrt{11} \simeq 3 + \frac{4}{3}\left(\frac{1}{4}\right) - \frac{8}{27}\left(\frac{1}{4}\right)^2 \simeq \boxed{\frac{179}{54}}$$

Q2a) $\underline{\underline{x=2}}$: $y = \boxed{2e^{-1}} = \frac{2}{e}$

b) $h = \frac{b-a}{n} = \frac{4-0}{4} = 1$ //

$$\therefore \text{Area} \simeq \frac{1}{2}\times 1\left[0 + 4e^{-2} + 2\left(e^{-\frac{1}{2}} + 2e^{-1} + 3e^{-\frac{3}{2}}\right)\right]$$

$$\simeq \boxed{2.28}$$

ci) $\int \left(xe^{-\frac{1}{2}x}\right) dx$ ~ By parts : $\frac{dv}{dx} = e^{-\frac{1}{2}x}$

$v = -2e^{-\frac{1}{2}x}$

$u = x$

$u' = 1$

$$\Rightarrow \left[-2xe^{-\frac{1}{2}x}\right] + 2\int e^{-\frac{1}{2}x}\,dx$$

$$= \boxed{-2xe^{-\frac{1}{2}x} - 4e^{-\frac{1}{2}x} + c}$$

ii) $R = \left[-2xe^{-\frac{1}{2}x} - 4e^{-\frac{1}{2}x}\right]_0^4$

$= \left[-8e^{-2} - 4e^{-2}\right] - \left[-4\right]$

$= \boxed{4 - 12e^{-2}}$

Q3a) $x = 2t + 5$ $\qquad$ $y = 3 + \frac{4}{t}$

$\frac{dx}{dt} = 2$ $\qquad$ $\frac{dy}{dt} = -\frac{4}{t^2}$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\frac{4}{t^2}}{2} = -\frac{2}{t^2}$ //

<u>at (9,5)</u>: $x = 9 = 2t + 5$

$4 = 2t$

$t = 2$ //

$\therefore \frac{dy}{dx} = \frac{-2}{4} = \boxed{-\frac{1}{2}}$

b) $x - 5 = 2t$

$\frac{x-5}{2} = t$ // $\longrightarrow$ $y = 3 + \frac{4}{\frac{x-5}{2}}$

$y = \frac{3(x-5)}{(x-5)} + \frac{8}{(x-5)}$

$$\Rightarrow \quad y = \frac{3(x-5)+8}{x-5} = \frac{3x-15+8}{x-5}$$

$$\Rightarrow \quad \boxed{y = \frac{3x-7}{x-5}}$$

Q4a) $\underline{\mu = 2}$: $\quad A = \begin{pmatrix} -9+10 \\ 8-8 \\ 5-6 \end{pmatrix} = \boxed{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}$

b) $\overrightarrow{OA} \cdot l_1$: $\quad \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -4 \\ -3 \end{pmatrix} = 5+3 = 8$ //

$$|\overrightarrow{OA}| = \sqrt{1^2+1^2} = \sqrt{2}$$

$$|\overrightarrow{l_1}| = \sqrt{3^2+4^2+5^2} = 5\sqrt{2}$$

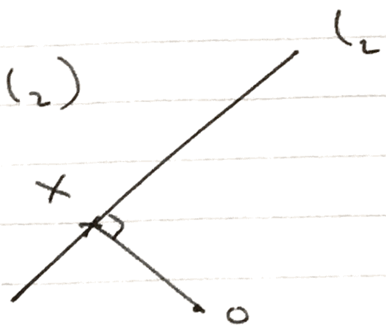
$$\therefore \cos\theta = \frac{8}{5\sqrt{2}\times\sqrt{2}} = \frac{8}{10} = \boxed{\frac{4}{5}}$$

c) $\overrightarrow{OB} = 3\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix}$

$$\therefore \quad \boxed{r_2 = \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix} + t\begin{pmatrix} 5 \\ -4 \\ -3 \end{pmatrix}}$$

d) $|\overrightarrow{OB}| = \sqrt{3^2+3^2} = \boxed{3\sqrt{2}}$

4e) $\overrightarrow{OX} = \begin{pmatrix} 3+5t \\ -4t \\ -3t-3 \end{pmatrix}$ (lies on l_2)



$\left(\overrightarrow{OX}\right) \cdot \begin{pmatrix} 5 \\ -4 \\ -3 \end{pmatrix} = 0$

since $\overrightarrow{OX}$ is $\perp$ to l_2.

$\Rightarrow \begin{pmatrix} 3+5t \\ -4t \\ -3t-3 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -4 \\ -3 \end{pmatrix} = 0$

$\Rightarrow 15 + 25t + 16t + 9t + 9 = 0$

$\Rightarrow 50t = -24$

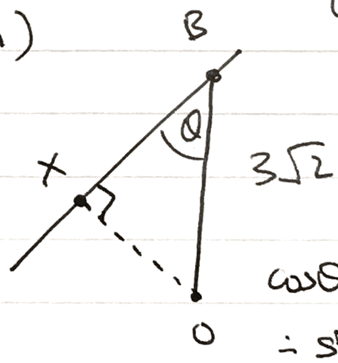
$\Rightarrow t = \frac{-12}{25} \longrightarrow \overrightarrow{OX} = \begin{pmatrix} 3 + 5\left(\frac{-12}{25}\right) \\ -4\left(\frac{-12}{25}\right) \\ -3\left(\frac{-12}{25}\right) - 3 \end{pmatrix}$

$\Rightarrow \overrightarrow{OX} = \begin{pmatrix} 3/5 \\ 48/25 \\ -39/25 \end{pmatrix}$

$|\overrightarrow{OX}| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{48}{25}\right)^2 + \left(-\frac{39}{25}\right)^2} = \frac{9}{5}\sqrt{2}$

$= \boxed{2.55}$

alt (Trigonometric approach)



$\frac{OX}{3\sqrt{2}} = \sin\theta \left(= \frac{\text{opposite}}{\text{hypotenuse}} \right)$

$\cos\theta = \frac{4}{5}$

$\therefore \sin\theta = \frac{3}{5}$

$\therefore OX = 3\sqrt{2} \times \frac{3}{5} = \frac{9\sqrt{2}}{5}$

Q5a) $\frac{d}{dx}\left(\sin(\pi y) - y - x^2 y = -5 \right)$

$$\Rightarrow \pi\cos(\pi y)\frac{dy}{dx} - \frac{dy}{dx} - 2xy - x^2\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx}\left(\pi\cos(\pi y) - 1 - x^2\right) = 2xy$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{2xy}{\pi\cos\pi y - x^2 - 1}}$$

b) at P, $\frac{dy}{dx} = \frac{2(2)(1)}{\pi\cos(\pi) - 4 - 1} = \frac{4}{-\pi - 5}$

$$= \frac{-4}{5+\pi} \quad //$$

$$\Rightarrow y - 1 = \frac{-4}{5+\pi}(x - 2)$$

$$\Rightarrow y = \frac{-4x}{\pi+5} + \frac{8}{\pi+5} + 1$$

<u>at x-axis, y=0</u> : $-4x + 8 + \pi + 5 = 0$

$$4x = \pi + 13$$

$$\boxed{x = \left(\frac{\pi + 13}{4}\right)}$$

Q6i a) $\dfrac{7x}{(x+3)(2x-1)} = \dfrac{A}{x+3} + \dfrac{B}{2x-1}$

$7x = A(2x-1) + B(x+3)$

$\underline{x=\frac{1}{2}}$: $\dfrac{7}{2} = \dfrac{7B}{2}$ $\therefore$ $B = 1$

$\underline{x=0}$: $0 = -A + 3B$ $\therefore$ $A = 3$

$\Rightarrow \dfrac{7x}{(x+3)(2x-1)} = \boxed{\dfrac{3}{x+3} + \dfrac{1}{2x-1}}$

b) $\displaystyle\int\left(\frac{3}{x+3} + \frac{1}{2x-1}\right)dx = 3\ln(x+3) + \frac{1}{2}\ln|2x-1|$ $(+c)$

$= \ln\left[(x+3)^3\right] + \ln\sqrt{2x-1} + c$

$= \boxed{\ln\left[(x+3)^3\sqrt{2x-1}\right] + c}$

ii) $\int \left(\frac{1}{x+x^{\frac{1}{3}}}\right) dx$

$$\left[\begin{array}{l} u^3 = x \\ 3u^2 \frac{du}{dx} = 1 \\ dx = 3u^2\, du \end{array}\right]$$

?

$\Rightarrow \int \frac{1}{u^3+u}\, du\,(3u^2) = 3\int \frac{u^2}{u(u^2+1)}\, du$

$\Rightarrow 3\int \frac{u}{u^2+1}\, du = \left[\frac{3}{2}\ln[u^2+1]\right] + c$

By pattern!

$\Rightarrow \boxed{\frac{3}{2}\ln\left(x^{\frac{2}{3}}+1\right) + c}$ $\quad \left(u^2 = x^{2/3}\right)$

Q7a) $V = \pi \int \left(y^2 \frac{dx}{d\theta}\right) d\theta$ $\quad x = \tan\theta$, $\frac{dx}{d\theta} = \sec^2\theta$.

$y^2 = (1+2\cos 2\theta)^2 = 1 + 4\cos 2\theta + 4\cos^2 2\theta$.

$\therefore y^2 \frac{dx}{d\theta} = \left[1 + 4(2\cos^2\theta - 1) + 4(2\cos^2\theta - 1)^2\right]\sec^2\theta$

$= \left[1 + 8\cos^2\theta - 4 + 4(4\cos^4\theta - 4\cos^2\theta + 1)\right]\sec^2\theta$

$= \left[16\cos^4\theta - 8\cos^2\theta + 1 - 4 + 4\right]\sec^2\theta$

$$= \frac{\left[16\cos^4\theta - 8\cos^2\theta + 1\right]}{\cos^2\theta}$$

$$= 16\cos^2\theta - 8 + \sec^2\theta = y^2\frac{dx}{d\theta}$$ //

hence $V = \pi \int_{\pi/4}^{\pi/3} \left[16\cos^2\theta - 8 + \sec^2\theta\right] d\theta$

Limits:

x	θ
1	$\pi/4$
$\sqrt{3}$	$\pi/3$

$$\left[\begin{array}{l} x = 1, \quad \theta = \tan^{-1}(1) = \frac{\pi}{4} \\ x = \sqrt{3}, \quad \theta = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3} \end{array}\right]$$

b) $V = \pi \int_{\pi/4}^{\pi/3} \left[16\left(\frac{1}{2}\cos 2\theta + \frac{1}{2}\right) - 8 + \sec^2\theta\right] d\theta$

$$\left[\begin{array}{l} \cos 2\theta = 2\cos^2\theta - 1 \\ \frac{\cos 2\theta + 1}{2} = \cos^2\theta \end{array}\right]$$

$$V = \pi \int_{\pi/4}^{\pi/3} \left[8\cos 2\theta + \sec^2\theta\right] d\theta$$

$$= \pi\left[4\sin 2\theta + \tan\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \pi\left[4\sin\frac{2\pi}{3} + \tan\frac{\pi}{3}\right]$$

$$-\pi\left[4\sin\frac{\pi}{2} + \tan\frac{\pi}{4}\right]$$

$$= \pi\left[3\sqrt{3}\right] - \pi\left[5\right] = \boxed{\pi\left[3\sqrt{3} - 5\right]}$$

Q8a) $\frac{dV}{dt} = -32\pi\sqrt{h}$ (liquid is leaving the tank).

$$\boxed{\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}}$$

$V = \pi r^2 h = \pi (40)^2 h$

$V = 1600\pi h$

$\therefore \frac{dV}{dh} = 1600\pi \quad \therefore \frac{dh}{dV} = \frac{1}{1600\pi}$.

$$\Rightarrow \frac{dh}{dt} = -32\cancel{\pi}\sqrt{h} \times \frac{1}{1600\cancel{\pi}} = \frac{-32\sqrt{h}}{1600}$$

$$\Rightarrow \frac{dh}{dt} = -0.02\sqrt{h}$$

□

b) $(h^{-\frac{1}{2}})\frac{dh}{dt} = -0.02$

$$\int (h^{-\frac{1}{2}})\,dh = \int (-0.02)\,dt$$

$2\sqrt{h} = -0.02t + c$.

let $t=0,\ h=100$: $20 = c$

$$\Rightarrow \frac{2\sqrt{h} - 20}{-0.02} = t$$

$h = 50$, $t = \frac{2\sqrt{50} - 20}{-0.02} = \boxed{293 \text{ min}}$