C4 June 2008

Q1

$y = e^{0.5x^2}$

(a)

x	0	0.4	0.8	1.2	1.6	2
y	e^0	$e^{0.08}$	$e^{0.32}$	$e^{0.72}$	$e^{1.28}$	e^2

(b)

$$R = I \approx \frac{h}{2}\left[y_0 + y_n + 2(y_1 + \dots + y_{n-1}) \right]$$

$$= \frac{0.4}{2}\left[e^0 + e^2 + 2\left(e^{0.08} + e^{0.32} + e^{0.72} + e^{1.28}\right)\right]$$

$$\underline{\underline{= 4.922}} \text{ (4sf)}$$

Q2

(a) $I_1 = \int x e^x \, dx$

Integrating by parts:

$u = x \qquad \frac{dv}{dx} = e^x$

$\frac{du}{dx} = 1 \qquad v = e^x$

$$I_1 = xe^x - \int e^x dx$$

$$\underline{= xe^x - e^x + C}$$

(b) $I_2 = \int x^2 e^x \, dx$

Integrating by parts:

$u = x^2 \qquad \frac{dv}{dx} = e^x$

$\frac{du}{dx} = 2x \qquad v = e^x$

$$I_2 = x^2 e^x - \int 2xe^x dx$$

$$= x^2 e^x - 2I_1$$

$$= x^2 e^x - 2(xe^x - e^x) + C$$

$$= \underline{x^2 e^x - 2xe^x + 2e^x} + C$$

Q3

Volume $= \pi x^2 \times 5x = 5\pi x^3$

(a) Area of the cross section: $A = \pi x^2 \Rightarrow \frac{dA}{dx} = 2\pi x$

$$\frac{dA}{dt} = 0.032$$

$$\frac{dA}{dt} = \frac{dA}{dx} \cdot \frac{dx}{dt}$$

$$0.032 = 2\pi x \cdot \frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{0.032}{2\pi x}$$

When $x = 2$cm $\quad \frac{dx}{dt} = \frac{0.032}{2\pi \times 2}$

$$= \frac{1}{125\pi}$$

$$= 0.00255 \text{ cm/s}$$

(b) $\frac{dV}{dt} = \frac{dV}{dx} \cdot \frac{dx}{dt} \qquad \frac{dV}{dx} = \frac{d(5\pi x^3)}{dx} = 15\pi x^2$

$$= 15\pi x^{\cancel{2}} \cdot \frac{0.032}{2\pi \cancel{x}}$$

$$= 0.24x$$

When $x = 2$cm $\Rightarrow \frac{dV}{dt} = 0.24 \times 2 = 0.48 \text{ cm}^3/\text{s}$

Q4

$$3x^2 - y^2 + xy = 4$$

(a) Differentiating implicitly with respect to x:

$$6x - 2y\frac{dy}{dx} + \left(y + x\frac{dy}{dx}\right) = 0$$

$$\frac{dy}{dx}(x - 2y) = -6x - y$$

$$\frac{dy}{dx} = \frac{-6x-y}{x-2y}$$

$\frac{dy}{dx} = \frac{8}{3}$ at P and at Q

So. $\frac{8}{3} = \frac{-6x-y}{x-2y}$

$$8(x-2y) = 3(-6x-y)$$

$$8x - 16y = -18x - 3y$$

$$26x = 13y$$

$$2x = y$$

So $y - 2x = 0$

(b) Find the coordinates of P and Q:

$y - 2x = 0$ and $3x^2 - y^2 + xy = 4$

$y = 2x \xrightarrow{\text{substitute}} 3x^2 - 4x^2 + x(2x) = 4$

$x^2 = 4$ $\quad$ P: $x = -2$, $y = -4$ or Q: $x = 2$, $y = 4$

Q5

(a) $\frac{1}{\sqrt{(4-3x)}} = (4-3x)^{-\frac{1}{2}}$

$$= 4^{-\frac{1}{2}}\left(1 - \frac{3}{4}x\right)^{-\frac{1}{2}}$$

$$= \frac{1}{2}\left(1 - \frac{3}{4}x\right)^{-\frac{1}{2}}$$

$$\left(1 - \frac{3}{4}x\right)^{-\frac{1}{2}} =$$

$$= 1 + \left(-\frac{1}{2}\right)\left(-\frac{3}{4}x\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}\left(-\frac{3}{4}x\right)^2 + \dots$$

$$= \left(1 + \frac{3}{8}x + \frac{27}{128}x^2 + \dots\right) \times \frac{1}{2}$$

$$\frac{1}{2} + \frac{3}{16}x + \frac{27}{256}x^2 + \dots$$

(b) $\frac{x+8}{\sqrt{4-3x}}$

$$= (x+8)(4-3x)^{-\frac{1}{2}}$$

$$\approx (x+8)\left(\frac{1}{2} + \frac{3}{16}x + \frac{27}{256}x^2 + \dots\right)$$

$$= \frac{\cancel{8}^4}{\cancel{2}_1} + \frac{\cancel{8}^1 \times 3}{\cancel{16}_2}x + \frac{\cancel{8}^1 \times 27}{\cancel{256}_{32}}x^2 + \dots$$

$$+ \quad \frac{1}{2}x + \frac{3^{\times 2}}{16_{\times 2}}x^2 + \frac{27}{256}x^3 + \dots$$

$\left(\frac{3}{2} + \frac{1}{2}\right)x \qquad \left(\frac{27}{32} + \frac{6}{32}\right)x^2$

$$= 4 + 2x + \frac{33}{32}x^2$$

Q6 $l_1: \underline{r}_1 = \begin{pmatrix}-9\\0\\10\end{pmatrix} + \lambda\begin{pmatrix}2\\1\\-1\end{pmatrix}$ $l_2: \underline{r}_2 = \begin{pmatrix}3\\1\\17\end{pmatrix} + \mu\begin{pmatrix}3\\-1\\5\end{pmatrix}$

(a) l_1 meet l_2 if $\underline{r_1 = r_2}$ for the same μ and λ

$$\begin{pmatrix}-9+2\lambda\\ \cancel{2}\lambda\\ 10-\lambda\end{pmatrix} = \begin{pmatrix}3+3\mu\\ 1-\mu\\ 17+5\mu\end{pmatrix} \begin{matrix}(i)\\(ii)\\(iii)\end{matrix}$$

From (i) and (ii)

(i) $-9+2\lambda = 3+3\mu$ $\quad 2\lambda = 12+3\mu$

(ii) $\lambda = 1-\mu$ $\quad \lambda = 1-\mu \quad \times 2$

$2\lambda = 2-2\mu$

So $2\lambda = 12+3\mu$

$\underline{2\lambda = 2-2\mu}$

$0 = 10+5\mu$

$\mu = -2$

$\lambda = 1--2 = 3$

Checking in eq (iii) $10-\lambda = 17+5\mu$

$10-3 = 17-10$ ✓

Hence they intersect at $(-3,3,7)$

$\lambda = 3,\ \mu = -2$ $\begin{pmatrix}3+3\mu\\1-\mu\\17+5\mu\end{pmatrix} = \begin{pmatrix}3-6\\1+2\\17-10\end{pmatrix} = \begin{pmatrix}-3\\3\\7\end{pmatrix}$

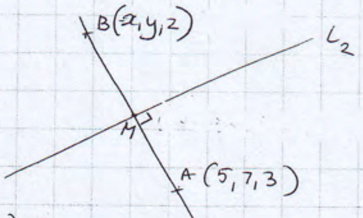
Q6 (c) $\underline{r}_{(A)} = 5i+7j+3k$

If A lies on l_1 then we can find a common λ for all 3 coordinates.

$[l_1]: \begin{pmatrix}-9+2\lambda\\ \lambda\\ 10-\lambda\end{pmatrix} = \begin{pmatrix}5\\7\\3\end{pmatrix} [A]$ $\quad -9+2\lambda = 5 \Rightarrow \lambda = 7$; $\lambda = 7$ ✓; $10-\lambda = 3 \Rightarrow \lambda = 7$

Hence A is on l_1

(d)



$M(-3,3,7)$ from part(a) l_1 is the point of intersection and a midpoint of AB

Let $B(x,y,z)$

Then midpoint $M = \left(\frac{x+5}{2}, \frac{y+7}{2}, \frac{z+3}{2}\right)$

$\frac{x+5}{2} = -3$ $\quad x = -11$

$\frac{y+7}{2} = 3$ $\quad y = -1$

$\frac{z+3}{2} = 7$ $\quad z = 11$

$B(-11,-1,11)$

Q6 (b) If l_1 and l_2 are perpendicular to each other than $\cos\theta$ of an angle between their direction vectors is equal 0.

Direction vectors of l_1 and l_2 are:

$\underline{d}_1 = \begin{pmatrix}2\\1\\-1\end{pmatrix}$ $\quad \underline{d}_2 = \begin{pmatrix}3\\-1\\5\end{pmatrix}$

From the dot product we get

$$\cos\theta = \frac{\underline{d}_1 \cdot \underline{d}_2}{|\underline{d}_1| \times |\underline{d}_2|} = \frac{\begin{pmatrix}2\\1\\-1\end{pmatrix}\cdot\begin{pmatrix}3\\-1\\5\end{pmatrix}}{\sqrt{2^2+1^2+(-1)^2} \times \sqrt{3^2+(-1)^2+5^2}}$$

$$= \frac{2\times3 + 1\times(-1) + (-1)\times5}{\sqrt{6}\times\sqrt{35}}$$

$$= \frac{6-1-5}{\sqrt{6}\times\sqrt{35}}$$

$$= 0$$

Hence $\cos\theta = 0$ so the angle θ between l_1 and l_2 must be $90°$.

Q7 (a) $\frac{2}{4-y^2} = \frac{2}{(2-y)(2+y)} \equiv \frac{A}{2-y} + \frac{B}{2+y}$

$$= \frac{A(2+y) + B(2-y)}{(2-y)(2+y)}$$

So, comparing the numerators:

$2 \equiv A(2+y) + B(2-y)$

Let $y=-2$, $2 \equiv B(2+2)$

$4B = 2$

$B = \frac{1}{2}$

Let $y = 2$, $2 = 4A$

$A = \frac{1}{2}$

$$\frac{2}{4-y^2} = \frac{1}{2(2-y)} + \frac{1}{2(2+y)}$$

(b) $2\cot x \frac{dy}{dx} = (4-y^2)$

By separating the variables we obtain:

$$\int \frac{2dy}{4-y^2} = \int \frac{dx}{\cot x}$$

$$\int\left(\frac{1}{2(2-y)} + \frac{1}{2(2+y)}\right)dy = \int \tan x\, dx$$

$\int \tan x\, dx = -\ln|\cos x| + c$

or $= \ln|\sec x|$

Q7 b)

$$\frac{1}{2}\left(-\ln|2-y|+\ln|2+y|\right)=-\ln|\cos x|+c$$

$x=\frac{\pi}{3}$ when $y=0$:

$$\frac{1}{2}\left(-\ln 2+\ln 2\right)=-\ln\left|\cos\frac{\pi}{3}\right|+c$$

$0=-\ln(\tfrac{1}{2})+c$

$c=\ln(\tfrac{1}{2})$

$c=-\ln 2$

So $\frac{1}{2}\left(-\ln|2-y|+\ln|2+y|\right)=-\ln|\cos x| \quad -\ln 2$

$$\ln\left|\frac{2+y}{2-y}\right|=-2\ln|\cos x|-2\ln 2$$

$$\ln\left|\frac{2+y}{2-y}\right|=\ln|\cos^{-2}x|-\ln 4$$

$$\ln\left|\frac{2+y}{2-y}\right|=\ln(\sec^2 x)-\ln 4$$

$$\frac{2+y}{2-y}=\frac{1}{4}\sec^2 x$$

$$\sec^2 x=\frac{4(2+y)}{2-y}$$

Q8

$x=8\cos t \quad y=4\sin 2t \quad 0\le t\le\frac{\pi}{2}$

$P(4,2\sqrt{3})$

a) $x=4 \Rightarrow 4=8\cos t$

$\cos t=\frac{1}{2}$

$\underline{t=\frac{\pi}{3}}$

$y=4\sin 2t$

$2\sqrt{3}=4\sin 2t$

$\sin 2t=\frac{2\sqrt{3}}{4}$

$\sin 2t=\frac{\sqrt{3}}{2} \quad 0\le 2t\le\pi$

$2t=\frac{\pi}{3},\frac{2\pi}{3}$

$t=\frac{\pi}{6},\underline{\frac{\pi}{3}}$

Hence $t=\frac{\pi}{3}$

(b) l is a normal at P

$$\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}$$

$$=-\frac{8\cos 2t}{8\sin t}$$

$$=-\frac{\cos 2t}{\sin t}\Bigg|_{t=\frac{\pi}{3}}$$

$$=-\frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}}$$

$$=\frac{1}{\sqrt{3}} \Leftarrow \text{gradient of tangent}$$

$\frac{dy}{dt}=(4\sin 2t)'$

$=8\cos 2t$

$\frac{dx}{dt}=(8\cos t)'$

$=-8\sin t$

$\cos\left(2\times\frac{\pi}{3}\right)=-\frac{1}{2}$

$\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$

gradient of the normal is $-\sqrt{3}$

Q8(b)

gradient $-\sqrt{3}$

point $(4,2\sqrt{3})$

Equation: $y-y_1=m(x-x_1)$

$y-2\sqrt{3}=-\sqrt{3}(x-4)$

$y=-\sqrt{3}x+4\sqrt{3}+2\sqrt{3}$

$\underline{y=-\sqrt{3}x+6\sqrt{3}}$

As required.

Q8(c)

$$\text{Area}=\int_0^4 y\,dx=\int_{t(0)}^{t(4)} y\frac{dx}{dt}\,dt \qquad \frac{dx}{dt}=-8\sin t$$

$$=\int_{\frac{\pi}{2}}^{\frac{\pi}{3}} 4\sin 2t\times-8\sin t\,dt$$

$$=-\int_{\frac{\pi}{2}}^{\frac{\pi}{3}} 32\times 2\sin t\cos t\times\sin t\,dt$$

$$=\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 64\sin^2 t\cos t\,dt$$

$t(4) \Rightarrow x=4$

$4=8\cos t$

$t=\frac{\pi}{3}$

$t(0) \Rightarrow x=0$

$8\cos t=0$

$\cos t=0$

$t=\frac{\pi}{2}$

Q8 (d)

$$I=\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 64\underbrace{\sin^2 t}_{u^2}\underbrace{\cos t\,dt}_{du}$$

Let $u=\sin t$

$\frac{du}{dt}=\cos t \qquad \boxed{du=\cos t\,dt}$

Substituting into the integral:

$$I=\int_{u(\frac{\pi}{3})}^{u(\frac{\pi}{2})} 64u^2\,du$$

Changing the limits

$u(\frac{\pi}{2})=\sin\frac{\pi}{2}=1$

$u(\frac{\pi}{3})=\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$

$$=\left[\frac{64}{3}u^3\right]_{\frac{\sqrt{3}}{2}}^{1}$$

$$=\frac{64}{3}-\frac{64}{3}\times\frac{\sqrt{3}}{2}\times\frac{3}{4}$$

$$=\frac{64}{3}-\frac{\cancel{32}^{\,8}}{\cancel{3}_{\,1}}\sqrt{3}\times\frac{\cancel{3}^{\,1}}{\cancel{4}_{\,1}}=\frac{64}{3}-8\sqrt{3}$$

As required