C4 June 2006 Model Solutions

(1) $C: \; 3x^2 - 2y^2 + 2x - 3y + 5 = 0$.

Differentiate w.r.t x:

$$6x - 4y\frac{dy}{dx} + 2 - 3\frac{dy}{dx} = 0$$
$$6x + 2 = (4y+3)\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{6x+2}{4y+3}$$

At $(0,1)$ $\frac{dy}{dx} = \frac{2}{7}$ $\therefore$ $m_{normal} = -\frac{7}{2}$

Equation of normal: $y - 1 = -\frac{7}{2}(x - 0)$
$$y - 1 + \tfrac{7}{2}x = 0$$
$$2y - 2 + 7x = 0.$$

(2) $f(x) = \dfrac{3x-1}{(1-2x)^2}$ $\quad |x| < \frac{1}{2}$

a) $\dfrac{3x-1}{(1-2x)^2} = \dfrac{A}{1-2x} + \dfrac{B}{(1-2x)^2}$

$$3x - 1 = A(1-2x) + B$$
$$3x - 1 = -2Ax + A + B$$

Comparing coefficients: x^1: $3 = -2A \Rightarrow \underline{A = -\frac{3}{2}}$

x^0: $-1 = A + B \Rightarrow \underline{B = \frac{1}{2}}$

b) $f(x) = \dfrac{1}{2(1-2x)^2} - \dfrac{3}{2(1-2x)} = \frac{1}{2}(1-2x)^{-2} - \frac{3}{2}(1-2x)^{-1}$

$$\tfrac{1}{2}(1-2x)^{-2} = \tfrac{1}{2}\left[1 + (-2)(-2x) + \frac{(-2)(-3)}{2!}(-2x)^2 + \frac{(-2)(-3)(-4)}{3!}(-2x)^3 + \dots\right]$$
$$= \tfrac{1}{2}\left[1 + 4x + 12x^2 + 32x^3 + \dots\right]$$
$$= \tfrac{1}{2} + 2x + 6x^2 + 16x^3 + \dots$$

$$\tfrac{3}{2}(1-2x)^{-1} = \tfrac{3}{2}\left[1 + (-1)(-2x) + \frac{(-1)(-2)}{2!}(-2x)^2 + \frac{(-1)(-2)(-3)}{3!}(-2x)^3 + \dots\right]$$
$$= \tfrac{3}{2}\left[1 + 2x + 4x^2 + 8x^3 + \dots\right]$$
$$= \tfrac{3}{2} + 3x + 6x^2 + 12x^3 + \dots$$

$$f(x) \approx (\tfrac{1}{2} + 2x + 6x^2 + 16x^3) - (\tfrac{3}{2} + 3x + 6x^2 + 12x^3)$$
$$\approx \underline{-1 - x + 4x^3}$$

(3) $y = 3\sin\frac{x}{2}$, $\quad 0 \le x \le 2\pi$

a) $A = \displaystyle\int_0^{2\pi} 3\sin(\tfrac{1}{2}x)\,dx$

$$= \left[-\frac{3\cos(\frac{1}{2}x)}{\frac{1}{2}}\right]_0^{2\pi} = \left[-6\cos\tfrac{x}{2}\right]_0^{2\pi}$$
$$= \left[(-6\cos\pi) - (-6\cos 0)\right]$$
$$= 6 - -6$$
$$= \underline{12}$$

b) $V = \displaystyle\int_0^{2\pi} \pi y^2\,dx$ $\qquad y^2 = 9\sin^2\frac{x}{2}$

$$= 9\pi\int_0^{2\pi} \sin^2\tfrac{x}{2}\,dx$$

$\cos 2A = 1 - 2\sin^2 A$

$\Rightarrow \sin^2 A = \frac{1}{2}(1 - \cos 2A)$

$\Rightarrow \sin^2\frac{x}{2} = \frac{1}{2}(1 - \cos x)$

$$= 9\pi\int_0^{2\pi} \tfrac{1}{2}(1 - \cos x)\,dx$$
$$= \frac{9\pi}{2}\int_0^{2\pi}(1 - \cos x)\,dx = \frac{9\pi}{2}\left[x - \sin x\right]_0^{2\pi}$$
$$= \frac{9\pi}{2}\left[(2\pi - \sin 2\pi) - (0 - \sin 0)\right]$$
$$= \frac{9\pi}{2}\left[2\pi\right]$$
$$= \underline{9\pi^2}$$

(4) $x = \sin t$ $\quad y = \sin(t + \frac{\pi}{6})$

a) At $t = \frac{\pi}{6}$, $x = \sin\frac{\pi}{6} = \frac{1}{2}$

$y = \sin(\frac{\pi}{6} + \frac{\pi}{6}) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

$\frac{dx}{dt} = \cos t$ $\qquad \frac{dy}{dt} = \cos(t + \frac{\pi}{6})$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \dfrac{\cos(t + \frac{\pi}{6})}{\cos t}$

At $t = \frac{\pi}{6}$, $\frac{dy}{dx} = \dfrac{\cos\frac{\pi}{3}}{\cos\frac{\pi}{6}} = \dfrac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$

Equation of tangent: $y - \frac{\sqrt{3}}{2} = \frac{1}{\sqrt{3}}(x - \frac{1}{2})$

$(\times 2\sqrt{3})$ $\quad 2\sqrt{3}y - 3 = 2(x - \frac{1}{2})$

$$\underline{2\sqrt{3}y - 2x - 2 = 0}$$

(4) b) $x = \sin t$ $\quad y = \sin(t + \frac{\pi}{6})$
$$= \sin t\cos\tfrac{\pi}{6} + \cos t\sin\tfrac{\pi}{6}$$
$$= \tfrac{\sqrt{3}}{2}\sin t + \tfrac{1}{2}\cos t$$

If $x = \sin t$, $x^2 = \sin^2 t \Rightarrow \cos t = \sqrt{1 - \sin^2 t} = \sqrt{1 - x^2}$

So $y = \frac{\sqrt{3}}{2}\sin t + \frac{1}{2}\cos t$
$$= \tfrac{\sqrt{3}}{2}x + \tfrac{1}{2}\sqrt{1 - x^2} \qquad \text{QED.}$$

(5) L_1: $\mathbf{r} = \begin{pmatrix}6\\19\\-1\end{pmatrix} + \lambda\begin{pmatrix}1\\4\\-2\end{pmatrix}$

a) $\begin{pmatrix}0\\a\\b\end{pmatrix} = \begin{pmatrix}6\\19\\-1\end{pmatrix} + \begin{pmatrix}\lambda\\4\lambda\\-2\lambda\end{pmatrix} \Rightarrow 0 = 6 + \lambda \Rightarrow \lambda = -6$

$a = 19 + 4\times -6 = 19 - 24 = -5$

$b = -1 + -2\times -6 = 11$.

$\underline{a = -5}$, $\underline{b = 11}$ $\quad \overrightarrow{OA} = \begin{pmatrix}0\\-5\\11\end{pmatrix}$

b) $\overrightarrow{OP} = \begin{pmatrix}6\\19\\-1\end{pmatrix} + \begin{pmatrix}\lambda\\4\lambda\\-2\lambda\end{pmatrix} = \begin{pmatrix}6+\lambda\\19+4\lambda\\-1-2\lambda\end{pmatrix}$

$\overrightarrow{OP}\cdot \mathbf{d}_1 = 0$ $\quad$ (where $\mathbf{d}_1$ = direction vector of L_1).

$\begin{pmatrix}6+\lambda\\19+4\lambda\\-1-2\lambda\end{pmatrix}\cdot\begin{pmatrix}1\\4\\-2\end{pmatrix} = 0 \Rightarrow 6 + \lambda + 76 + 16\lambda + 2 + 4\lambda = 0$

$\Rightarrow 84 + 21\lambda = 0 \Rightarrow \lambda = -4$.

$\overrightarrow{OP} = \begin{pmatrix}6-4\\19+4\times -4\\-1-2\times -4\end{pmatrix} = \underline{\begin{pmatrix}2\\3\\7\end{pmatrix}}$

c) $\overrightarrow{OB} = \begin{pmatrix}5\\15\\1\end{pmatrix}$ $\quad$ Does B lie on L_1? $\begin{pmatrix}5\\15\\1\end{pmatrix} = \begin{pmatrix}6+\lambda\\19+4\lambda\\-1-2\lambda\end{pmatrix} \begin{array}{l}\Rightarrow \lambda = -1\\ \Rightarrow \lambda = -1\\ \Rightarrow \lambda = -1\end{array}$ ✓

So A, P and B lie on L_1, hence are colinear.

$\overrightarrow{AP} = \mathbf{p} - \mathbf{a} = \begin{pmatrix}2\\3\\7\end{pmatrix} - \begin{pmatrix}0\\-5\\11\end{pmatrix} = \begin{pmatrix}2\\8\\-4\end{pmatrix}$

$\overrightarrow{PB} = \mathbf{b} - \mathbf{p} = \begin{pmatrix}5\\15\\1\end{pmatrix} - \begin{pmatrix}2\\3\\7\end{pmatrix} = \begin{pmatrix}3\\12\\-6\end{pmatrix}$

$AP = \sqrt{2^2 + 8^2 + 4^2} = \sqrt{84} = 2\sqrt{21}$

$PB = \sqrt{3^2 + 12^2 + 6^2} = \sqrt{189} = 3\sqrt{21}$

$AP : PB = 2\sqrt{21} : 3\sqrt{21}$
$$= \underline{2 : 3}$$

(6) $y = (x-1)\ln x$, $\quad x > 0$.

a)

x	1	1.5	2	2.5	3
y	0	$\frac{1}{2}\ln\frac{3}{2}$	$\ln 2$	$\frac{3}{2}\ln\frac{5}{2}$	$2\ln 3$

b) $I = \displaystyle\int_1^3 (x-1)\ln x\,dx$

i) 2 strips: $n = 2$, $h = 1$

$I \approx \frac{1}{2}\times 1\{0 + 2\ln 3 + 2\times\ln 2\}$

$\approx \frac{1}{2}(2\ln 3 + 2\ln 2)$

$\approx \underline{1.792}$ $\quad$ (4 s.f.)

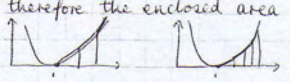
ii) 4 strips: $n = 4$, $h = 0.5$

$I \approx \frac{1}{2}\times 0.5\{0 + 2\ln 3 + 2(\frac{1}{2}\ln\frac{3}{2} + \ln 2 + \frac{3}{2}\ln\frac{5}{2})\}$

$\approx \frac{1}{4}(2\ln 3 + \ln\frac{3}{2} + 2\ln 2 + 3\ln\frac{5}{2})$

$\approx \underline{1.684}$ $\quad$ (4 s.f.)

c) The answer to part i) is an overestimate. By increasing the number of values we increase the number of trapezia used. These are thinner and therefore the enclosed area is closer to the actual area.



d) $\displaystyle\int_1^3 (x-1)\ln x\,dx = \int_1^3 x\ln x - 1\ln x\,dx$

$$= \int_1^3 x\ln x\,dx - \int_1^3 1\ln x\,dx$$

Parts: $\int uv'\,dx = uv - \int vu'\,dx$

$$= \left[\tfrac{x^2}{2}\ln x - \int \tfrac{x^2}{2}\times\tfrac{1}{x}\,dx\right]_1^3 - \left[x\ln x - \int x\times\tfrac{1}{x}\,dx\right]$$
$$= \left[\tfrac{x^2}{2}\ln x - \tfrac{1}{2}\int x\,dx\right]_1^3 - \left[x\ln x - \int 1\,dx\right]_1^3$$
$$= \left[\tfrac{x^2}{2}\ln x - \tfrac{x^2}{4}\right]_1^3 - \left[x\ln x - x\right]_1^3$$
$$= \left[\tfrac{x^2}{2}\ln x - \tfrac{x^2}{4} - x\ln x + x\right]_1^3$$
$$= \left[(\tfrac{3^2}{2}\ln 3 - \tfrac{3^2}{4} - 3\ln 3 + 3) - (\tfrac{1}{2}\ln 1 - \tfrac{1}{4} - 1\ln 1 + 1)\right]$$

$(\ln 1 = 0)$
$$= \tfrac{9}{2}\ln 3 - \tfrac{9}{4} - 3\ln 3 + 3 + \tfrac{1}{4} - 1$$
$$= \tfrac{9}{2}\ln 3 - 3\ln 3$$
$$= \underline{\tfrac{3}{2}\ln 3} \qquad \text{QED.}$$