C4 W14R

1. (a) Find the binomial expansion of

$$\frac{1}{(4+3x)^3}, \qquad |x| < \frac{4}{3}$$

in ascending powers of x, up to and including the term in x^3.
Give each coefficient as a simplified fraction.

(6)

In the binomial expansion of

$$\frac{1}{(4-9x)^3}, \qquad |x| < \frac{4}{9}$$

the coefficient of x^2 is A.

(b) Using your answer to part (a), or otherwise, find the value of A.
Give your answer as a simplified fraction.

(2)

a) $(4+3x)^{-3} = 4^{-3}(1+\frac{3}{4}x)^{-3} = \frac{1}{64}(1+\frac{3}{4}x)^{-3}$

$= \frac{1}{64}\left(1+(-3)(\frac{3}{4}x) + \frac{(-3)(-4)}{2}(\frac{3}{4}x)^2 + \frac{(-3)(-4)(-5)}{6}(\frac{3}{4}x)^3\right)$

$= \frac{1}{64} - \frac{9}{256}x + \frac{27}{512}x^2 - \frac{135}{2048}x^3$

b) replace $3x$ with $-9x$ in x^2 part of expansion

$\frac{1}{64}\left[\frac{(-3)(-4)}{2}\left(\frac{-9x}{4}\right)^2\right] = \frac{243}{512}x^2 \quad \therefore A = \frac{243}{512}$

PMT

2. (i) Find

$$\int x\cos\left(\frac{x}{2}\right)\mathrm{d}x$$

(3)

(ii) (a) Express $\frac{1}{x^2(1-3x)}$ in partial fractions.

(4)

(b) Hence find, for $0 < x < \frac{1}{3}$

$$\int \frac{1}{x^2(1-3x)}\mathrm{d}x$$

(3)

i) $u = x \quad v = 2\sin(\tfrac{1}{2}x) \qquad \int uv' = uv - \int u'v$

$u' = 1 \quad v' = \cos(\tfrac{1}{2}x)$

$= 2x\sin(\tfrac{1}{2}x) - \int 2\sin(\tfrac{1}{2}x)\,dx$

$= 2x\sin(\tfrac{1}{2}x) + 4\cos(\tfrac{1}{2}x) + C$

ii) $\frac{1}{x^2(1-3x)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{1-3x} \quad \therefore 1 = A(x)(1-3x) + B(1-3x) + C(x^2)$

$x = 0 \Rightarrow \underline{B=1} \quad x = \frac{1}{3} \Rightarrow \frac{1}{9}C = 1 \therefore \underline{C=9}$

$x = 1 \Rightarrow 1 = -2A - 2B + C \quad 2A = 9 - 2 - 1 = 6 \therefore \underline{A=3}$

$\Rightarrow \frac{3}{x} + \frac{1}{x^2} + \frac{9}{1-3x}$

b) $3\int \frac{1}{x}dx + \int x^{-2}dx - 3\int \frac{-3}{1-3x}dx$

$= 3\ln x - \frac{1}{x} - 3\ln(1-3x) + c$

$= 3\ln\left(\frac{x}{1-3x}\right) - \frac{1}{x} + c$

PMT

3. The number of bacteria, N, present in a liquid culture at time t hours after the start of a scientific study is modelled by the equation

$$N = 5000(1.04)^t, \qquad t \geqslant 0$$

where N is a continuous function of t.

(a) Find the number of bacteria present at the start of the scientific study. **(1)**

(b) Find the percentage increase in the number of bacteria present from $t = 0$ to $t = 2$ **(2)**

Given that $N = 15000$ when $t = T$,

(c) find the value of $\frac{dN}{dt}$ when $t = T$, giving your answer to 3 significant figures. **(4)**

a) $t=0 \Rightarrow N = 5000$

b) $t=2 \Rightarrow N = 5408 \Rightarrow \%$ Increase $= \frac{408}{50\cancel{00}} \times 1\cancel{00}$
$= 8.16\%$

c) $15000 = 5000(1.04)^T \Rightarrow 3 = 1.04^T$

$\ln(3) = T\ln(1.04) \Rightarrow T = 28.01102276\ldots$

$\frac{dN}{dt} = 5000(1.04)^t \times \ln(1.04)$

when $t=T$ $\frac{dN}{dt} = 588.3106\ldots \approx \underline{588}$ (3sf)

b) Area $\approx \frac{1}{2}(\ln 2)\left[2.1333+2(1.4792+1.0079)+0.6667\right]$

≈ 2.69

c) $u=1+3e^{-x} \Rightarrow \frac{du}{dx}=-3e^{-x} \Rightarrow -\frac{1}{3}du = e^{-x}dx$

$\Rightarrow \frac{4}{3}\int \frac{1}{u^{\frac{1}{2}}} \times -\frac{1}{3}\,du = \frac{-4}{9}\int u^{-\frac{1}{2}}du = \frac{-4}{9}\left[2u^{\frac{1}{2}}\right]$

$\Rightarrow -\frac{8}{9}\sqrt{1+3e^{-x}} + C$

$\int_{-3\ln 2}^{0} \frac{4e^{-x}}{3\sqrt{1+3e^{-x}}}dx = -\frac{8}{9}\left[\sqrt{1+3e^{-x}}\right]_{-3\ln 2}^{0}$

$= -\frac{8}{9}\left[(\sqrt{4})-(\sqrt{1+3e^{3\ln 2}})\right]$

$= -\frac{8}{9}\left[2-\sqrt{1+3\times 2^3}\right] = -\frac{8}{9}\left[2-5\right] = \frac{8}{3}$

4.

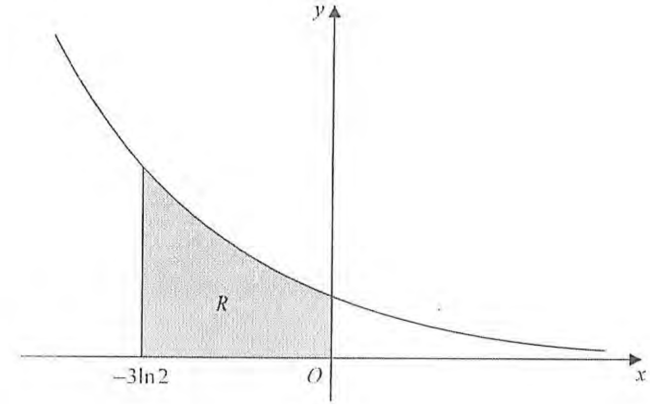


Figure 1

Figure 1 shows a sketch of part of the curve with equation $y = \frac{4e^{-x}}{3\sqrt{(1+3e^{-x})}}$

The finite region R, shown shaded in Figure 1, is bounded by the curve, the x-axis, the line $x = -3\ln 2$ and the y-axis.

The table below shows corresponding values of x and y for $y = \frac{4e^{-x}}{3\sqrt{(1+3e^{-x})}}$

x	$-3\ln 2$	$-2\ln 2$	$-\ln 2$	0
y	2.1333	1.4792	1.0079	0.6667

(a) Complete the table above by giving the missing value of y to 4 decimal places. **(1)**

(b) Use the trapezium rule, with all the values of y in the completed table, to obtain an estimate for the area of R, giving your answer to 2 decimal places. **(3)**

(c) (i) Using the substitution $u = 1 + 3e^{-x}$, or otherwise, find

$$\int \frac{4e^{-x}}{3\sqrt{(1+3e^{-x})}}\,dx$$

(5)

(ii) Hence find the value of the area of R.

(2)

5. Given that $y = 2$ at $x = \dfrac{\pi}{8}$, solve the differential equation

$$\frac{dy}{dx} = \frac{3y^2}{2\sin^2 2x}$$

giving your answer in the form $y = f(x)$.

(6)

$$\int \frac{1}{y^2}\,dy = \int \frac{3}{2}\operatorname{cosec}^2 2x$$

$$-\frac{1}{y} = -\frac{3}{4}\cot 2x + c$$

$$\therefore \quad \frac{1}{y} = \frac{3}{4} \times \frac{1}{\tan 2x} + c$$

$y = 2,\ x = \frac{\pi}{8}$ $\quad \frac{1}{2} = \frac{3}{4} + c \quad \therefore\ c = -\frac{1}{4}$

$$\frac{1}{y} = \frac{3}{4\tan 2x} - \frac{1}{4}\,\frac{\times \tan 2x}{\times \tan 2x} = \frac{3 - \tan 2x}{4\tan 2x}$$

$$\therefore \quad y = \frac{4\tan 2x}{3 - \tan 2x}$$

6. Oil is leaking from a storage container onto a flat section of concrete at a rate of $0.48 \text{ cm}^3 \text{s}^{-1}$. The leaking oil spreads to form a pool with an increasing circular cross-section. The pool has a constant uniform thickness of 3 mm.

Find the rate at which the radius r of the pool of oil is increasing at the instant when $r = 5$ cm. Give your answer, in cm s^{-1}, to 3 significant figures.

(5)

∴ Volume of cylinder Increases at a rate of 0.48 cm³/s

$\frac{dV}{dt} = 0.48 \qquad V = \pi r^2 \times 0.3 \Rightarrow \frac{dV}{dr} = 0.6\pi r$

$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt} = \frac{1}{0.6\pi r} \times 0.48$

$r = 5 \Rightarrow \frac{dr}{dt} = \frac{0.48}{3\pi} = 0.0509$ (3sf)

$\therefore \cos t = \frac{5}{6}$ $\cos t = -\frac{1}{2}$ $t = \frac{2\pi}{3}$ (as previous)

$\therefore t = 0.5857^c$

$x = 2\cos t = \frac{5}{3}$

$y = \sqrt{3}\cos 2t$

$y = \sqrt{3}(2\cos^2 t - 1)$

$y = \sqrt{3}\left(2 \times \left(\frac{5}{6}\right)^2 - 1\right)$

$y = \frac{7\sqrt{3}}{18}$ $\therefore Q\left(\frac{5}{3}, \frac{7\sqrt{3}}{18}\right)$

Q6

7. The curve C has parametric equations

$$x = 2\cos t, \quad y = \sqrt{3}\cos 2t, \qquad 0 \leqslant t \leqslant \pi$$

where t is a parameter.

(a) Find an expression for $\frac{dy}{dx}$ in terms of t. **(2)**

The point P lies on C where $t = \frac{2\pi}{3}$

The line l is a normal to C at P.

(b) Show that an equation for l is

$$2x - 2\sqrt{3}y - 1 = 0$$

(5)

The line l intersects the curve C again at the point Q.

(c) Find the exact coordinates of Q.
You must show clearly how you obtained your answers. **(6)**

$\frac{dx}{dt} = -2\sin t$ $\frac{dy}{dt} = -2\sqrt{3}\sin 2t$

$\therefore \frac{dy}{dx} = \frac{-2\sqrt{3}\sin 2t}{-2\sin t} = \frac{\sqrt{3}(2\cancel{\sin}t\cos t)}{\cancel{\sin}t} = 2\sqrt{3}\cos t$

$t = \frac{2\pi}{3}$ $x = 2\cos\left(\frac{2\pi}{3}\right) = -1$ $y = \sqrt{3}\cos\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

$\frac{dy}{dx} = 2\sqrt{3}\cos\left(\frac{2\pi}{3}\right) = -\sqrt{3} = M_t$ $\therefore M_n = \frac{1}{\sqrt{3}}$

$\Rightarrow y + \frac{\sqrt{3}}{2} = \frac{1}{\sqrt{3}}(x+1)$ $\Rightarrow \sqrt{3}y + \frac{3}{2} = x + 1$ $(\times 2)$

$\Rightarrow 2x - 2\sqrt{3}y - 1 = 0$ #

c) $4\cos t - 6\cos 2t - 1 = 0$ $4\cos t - 6(2\cos^2 t - 1) - 1 = 0$

$-12\cos^2 t + 4\cos t + 5 = 0 \Rightarrow 12\cos^2 t - 4\cos t - 5 = 0$

$(6\cos t - 5)(2\cos t + 1) = 0$

8. With respect to a fixed origin O, the lines l_1 and l_2 are given by the equations

$$l_1 : \mathbf{r} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \qquad l_2 : \mathbf{r} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ -2 \\ 5 \end{pmatrix}$$

where λ and μ are scalar parameters.

(a) Find, to the nearest 0.1°, the acute angle between l_1 and l_2 **(3)**

The point A has position vector $\begin{pmatrix} 0 \\ 1 \\ 6 \end{pmatrix}$.

(b) Show that A lies on l_1 **(1)**

The lines l_1 and l_2 intersect at the point X.

(c) Write down the coordinates of X. **(1)**

(d) Find the exact value of the distance AX. **(2)**

The distinct points B_1 and B_2 both lie on the line l_2

Given that $AX = XB_1 = XB_2$

(e) find the area of the triangle AB_1B_2 giving your answer to 3 significant figures. **(3)**

Given that the x coordinate of B_1 is positive,

(f) find the exact coordinates of B_1 and the exact coordinates of B_2 **(5)**

a) $a = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ $b = \begin{pmatrix} 5 \\ -2 \\ 5 \end{pmatrix}$ $\theta = \cos^{-1}\left|\left(\frac{-5-4+5}{\sqrt{6}\sqrt{54}}\right)\right| \therefore \cos^{-1}\left(\frac{4}{18}\right)$

$\theta = \cos^{-1}\left(\frac{2}{9}\right) = \underline{77.2°}$

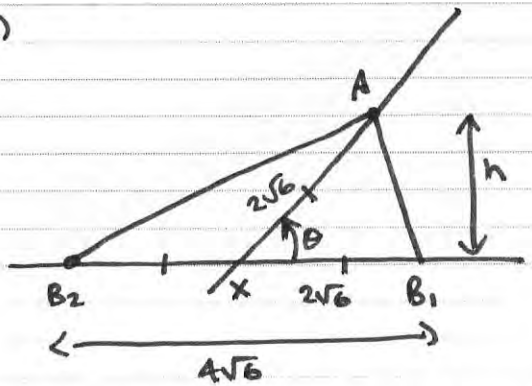
b) $r = \begin{pmatrix} 2-\lambda \\ -3+2\lambda \\ 4+\lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 6 \end{pmatrix}$ when $\lambda = 2$ $\therefore$ A lies on l_1

c) $X(2,-3,4)$

d) $\overrightarrow{AX} = x - a = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ -2 \end{pmatrix}$

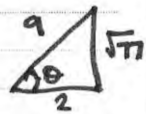
$\therefore |AX| = \sqrt{2^2+4^2+2^2} = \underline{2\sqrt{6}}$

e)



$h = 2\sqrt{6}\sin\theta$

$\cos\theta = \frac{2}{9}$



$\therefore \sin\theta = \frac{\sqrt{77}}{9}$

$h = 2\sqrt{6} \times \frac{\sqrt{77}}{9}$

$\text{Area} = \frac{1}{2}bh = 2\sqrt{6}\left(2\sqrt{6} \times \frac{\sqrt{77}}{9}\right)$

$= \frac{8\sqrt{77}}{3} \simeq \underline{23.4}$ (3sf)

f) $\overrightarrow{XB_1} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} + \mu\begin{pmatrix} 5 \\ -2 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} = \mu\begin{pmatrix} 5 \\ -2 \\ 5 \end{pmatrix}$

$\left|\begin{pmatrix} 5 \\ -2 \\ 5 \end{pmatrix}\right| = 3\sqrt{6}$ but $\overrightarrow{XB_1} = 2\sqrt{6}$ $\therefore \mu = \frac{2}{3}$

$B_1 = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 5 \\ -2 \\ 5 \end{pmatrix} = \begin{pmatrix} 16/3 \\ -13/3 \\ 22/3 \end{pmatrix}$ $B_2 = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 5 \\ -2 \\ 5 \end{pmatrix} = \begin{pmatrix} -4/3 \\ -5/3 \\ 2/3 \end{pmatrix}$