C4 JAN 11

1. Use integration to find the exact value of

$$\int_0^{\frac{\pi}{2}} x \sin 2x \, dx$$

(6)

$u = x \quad v' = \sin 2x \qquad \Rightarrow -\frac{1}{2}x\cos 2x + \frac{1}{2}\int \cos 2x \, dx$

$u' = 1 \quad v = -\frac{1}{2}\cos 2x$

$= \left[-\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{2}}$

$= \left(\frac{\pi}{4}\right) - (0) = \underline{\frac{\pi}{4}}$

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2. The current, I amps, in an electric circuit at time t seconds is given by

$$I = 16 - 16(0.5)^t, \qquad t \geqslant 0$$

Use differentiation to find the value of $\frac{dI}{dt}$ when $t = 3$.

Give your answer in the form $\ln a$, where a is a constant.

(5)

a) $\frac{dI}{dt} = -16\left(0.5^t \ln 0.5\right) = -16\left(0.5^t \times -\ln 2\right)$

$\frac{dI}{dt} = 16\ln 2 \times 0.5^t \quad t = 3 \Rightarrow \frac{dI}{dt} = 16\ln 2 \times \frac{1}{8}$

$= \underline{2\ln 2} = \underline{\ln 4}$

3. (a) Express $\dfrac{5}{(x-1)(3x+2)}$ in partial fractions.

(3)

(b) Hence find $\displaystyle\int \frac{5}{(x-1)(3x+2)}\,dx$, where $x>1$.

(3)

(c) Find the particular solution of the differential equation

$$(x-1)(3x+2)\frac{dy}{dx}=5y, \quad x>1,$$

for which $y=8$ at $x=2$. Give your answer in the form $y=f(x)$.

(6)

a) $5 = A(3x+2)+B(x-1) \Rightarrow$ $3A+B=0$, $2A-B=5$ (+): $5A=5 \Rightarrow A=1$, $B=-3$

$\Rightarrow \dfrac{1}{x-1}-\dfrac{3}{3x+2}$

b) $\ln(x-1)-\ln(3x+2)+C \Rightarrow \ln\left(\dfrac{x-1}{3x+2}\right)+C$

c) $\displaystyle\int \frac{1}{y}\,dy = \int \frac{5}{(x-1)(3x+2)}\,dx \Rightarrow \ln y = \ln\left(\frac{x-1}{3x+2}\right)+C$

$y=8, x=2 \Rightarrow \ln 8 = \ln\left(\frac{1}{8}\right)+C \Rightarrow \ln 8 = -\ln 8 + C$

$\Rightarrow C = 2\ln 8 = 6\ln 2 = \ln 64$

$\ln y = \ln\left(\dfrac{x-1}{3x+2}\right)+\ln 64$

$\ln y = \ln\left(\dfrac{64(x-1)}{3x+2}\right) \Rightarrow y = \dfrac{64(x-1)}{3x+2}$

4. Relative to a fixed origin O, the point A has position vector $\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ and the point B has position vector $-2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$. The points A and B lie on a straight line l.

(a) Find $\overrightarrow{AB}$.

(2)

(b) Find a vector equation of l.

(2)

The point C has position vector $2\mathbf{i} + p\mathbf{j} - 4\mathbf{k}$ with respect to O, where p is a constant. Given that AC is perpendicular to l, find

(c) the value of p,

(4)

(d) the distance AC.

(2)

a) $A\begin{pmatrix}1\\-3\\2\end{pmatrix} \quad B\begin{pmatrix}-2\\2\\-1\end{pmatrix} \quad \overrightarrow{AB} = \begin{pmatrix}-3\\5\\-3\end{pmatrix}$

b) $l = \begin{pmatrix}1\\-3\\2\end{pmatrix} + t\begin{pmatrix}-3\\5\\-3\end{pmatrix} = \begin{pmatrix}1-3t\\-3+5t\\2-3t\end{pmatrix}$

c) $\overrightarrow{AC} = \begin{pmatrix}2\\p\\-4\end{pmatrix} - \begin{pmatrix}1\\-3\\2\end{pmatrix} = \begin{pmatrix}1\\p+3\\-6\end{pmatrix}$

perp $\Rightarrow \overrightarrow{AC} \cdot \overrightarrow{AB} = 0 \quad \begin{pmatrix}-3\\5\\-3\end{pmatrix} \cdot \begin{pmatrix}1\\p+3\\-6\end{pmatrix} = 0$

$-3 + 5p + 15 + 18 = 0 \Rightarrow 5p = -30 \Rightarrow \underline{p = -6}$

d) $AC = \begin{pmatrix}1\\-3\\-6\end{pmatrix} \quad |AC| = \sqrt{1^2 + 3^2 + 6^2} = \underline{\sqrt{46}}$

5. (a) Use the binomial theorem to expand

$$(2-3x)^{-2}, \quad |x|<\frac{2}{3},$$

in ascending powers of x, up to and including the term in x^3. Give each coefficient as a simplified fraction.

(5)

$$f(x)=\frac{a+bx}{(2-3x)^2}, \quad |x|<\frac{2}{3}, \quad \text{where } a \text{ and } b \text{ are constants.}$$

In the binomial expansion of $f(x)$, in ascending powers of x, the coefficient of x is 0 and the coefficient of x^2 is $\frac{9}{16}$. Find

(b) the value of a and the value of b,

(5)

(c) the coefficient of x^3, giving your answer as a simplified fraction.

(3)

a) $2^{-2}\left(1-\frac{3}{2}x\right)^{-2} = \frac{1}{4}\left(1-\frac{3}{2}x\right)^{-2}$

$= \frac{1}{4}\left(1+(-2)(-\frac{3}{2}x) + \frac{(-2)(-3)}{2}\left(-\frac{3}{2}x\right)^2 + \frac{(-2)(-3)(-4)}{6}\left(-\frac{3}{2}x\right)^3\right)$

$= \frac{1}{4}\left(1+3x+\frac{27}{4}x^2+\frac{108}{8}x^3\right) = \frac{1}{4}+\frac{3}{4}x+\frac{27}{16}x^2+\frac{27}{8}x^3$

b) $f(x) = (a+bx)(2-3x)^{-2} \approx (a+bx)\left(\frac{1}{4}+\frac{3}{4}x+\frac{27}{16}x^2 \ldots\right)$

$\Rightarrow \frac{3}{4}a+\frac{1}{4}b=0 \quad (\times 4) \quad 3a+b=0$

$\frac{27}{16}a+\frac{3}{4}b=\frac{9}{16} \quad (\times 16) \quad 27a+12b=9$

$12a+4b=0$ −
$9a+4b=3$

$3a=-3 \Rightarrow a=-1 \quad b=3$

c) $\frac{27}{16}b+\frac{27}{8}a = \frac{81}{16}-\frac{27}{8} = \frac{27}{16}$

6. The curve C has parametric equations

$$x = \ln t, \quad y = t^2 - 2, \quad t > 0$$

Find

(a) an equation of the normal to C at the point where $t = 3$, **(6)**

(b) a cartesian equation of C. **(3)**

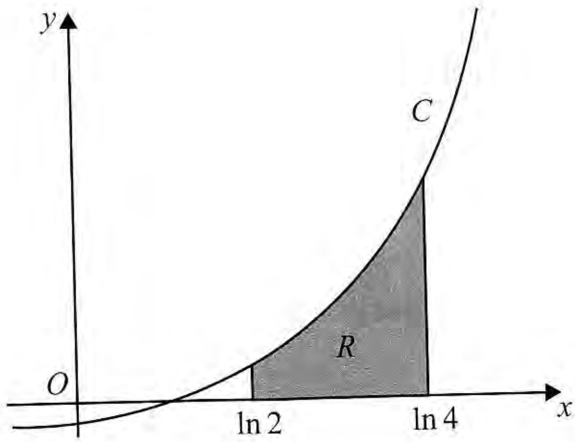

Figure 1

The finite area R, shown in Figure 1, is bounded by C, the x-axis, the line $x = \ln 2$ and the line $x = \ln 4$. The area R is rotated through $360°$ about the x-axis.

(c) Use calculus to find the exact volume of the solid generated. **(6)**

a) $\frac{dx}{dt} = \frac{1}{t}$ $\quad \frac{dy}{dt} = 2t$ $\quad \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = 2t \div \frac{1}{t}$

$= 2t^2$

$t = 3,\ x = \ln 3;\ y = 7;\ m_t = 18 \Rightarrow m_n = \frac{-1}{18}$

$y - 7 = \frac{-1}{18}(x - \ln 3)$

b) Volume $= \pi \int y^2 dx = \pi \int y^2 \frac{dx}{dt} dt$

$x = \ln 4 \quad t = 4$
$x = \ln 2 \quad t = 2$

Volume $= \pi \int_2^4 (t^2 - 2)^2 \times \frac{1}{t} dt$

$= \pi \int_2^4 t^3 - 4t + 4t^{-1} dt$

$$= \pi \left[\tfrac{1}{4}t^4 - 2t^2 + 4\ln t \right]_2^4$$

$$= \pi \left[(64 - 32 + 4\ln 4) - (4 - 8 + 4\ln 2) \right]$$

$$= \pi (36 + 4\ln 2) = 4\pi(9 + \ln 2)$$

7.

$$I = \int_2^5 \frac{1}{4+\sqrt{(x-1)}}\,dx$$

(a) Given that $y = \dfrac{1}{4+\sqrt{(x-1)}}$, complete the table below with values of y corresponding to $x = 3$ and $x = 5$. Give your values to 4 decimal places.

x	2	3	4	5
y	0.2	0.1847	0.1745	0.1667

(2)

(b) Use the trapezium rule, with all of the values of y in the completed table, to obtain an estimate of I, giving your answer to 3 decimal places.

(4)

(c) Using the substitution $x = (u-4)^2 + 1$, or otherwise, and integrating, find the exact value of I.

(8)

b) Area $\simeq \frac{1}{2}(1)\left[0.2 + 2(0.1847 + 0.1745) + 0.1667\right]$

$\simeq 0.543$ (3dp)

c) $x = (u-4)^2+1 = u^2 - 8u + 17$

$\frac{dx}{du} = 2u - 8 \Rightarrow dx = (2u-8)du$

$x - 1 = (u-4)^2 \Rightarrow \sqrt{x-1} = u - 4$

$\Rightarrow 4 + \sqrt{x-1} = u$

$x=5 \quad 5 = (u-4)^2+1 \quad u-4 = 2 \Rightarrow u = 6$

$x=2 \quad 2 = (u-4)^2+1 \quad u-4 = 1 \Rightarrow u = 5$

$= \int_5^6 \frac{2u-8}{u}\,du = \int_5^6 2 - 8u\,du = \left[2u - 8\ln u\right]_5^6$

$= (12 - 8\ln 6) - (10 - 8\ln 5) = 2 + 8\ln\frac{5}{6}$