CORE 4 JAN 06

Q1 $3x^2+4y^2-2x+6xy-5=0$

differentiating implicitly with respect to x:

$$6x+8y\frac{dy}{dx}-2+\left(6y+6x\frac{dy}{dx}\right)=0$$

$$\frac{dy}{dx}\left(8y+6x\right)=2-6x-6y$$

$$\frac{dy}{dx}=\frac{2-6x-6y}{8y+6x}=\frac{1-3x-3y}{4y+3x}$$

Tangent at $(1,-2)$:

$$m=\left.\frac{dy}{dx}\right|_{(1,-2)}=\frac{1-3\times1-3\times-2}{4\times-2+3\times1}=\frac{1-3+6}{-8+3}=\frac{4}{-5}$$

$$y-y_1=m(x-x_1)$$

$$y+2=-\frac{4}{5}(x-1)$$

$$5y+10=-4x+4$$

$$\underline{4x+5y+6=0}$$

Q2 (a) $y=\sec x$

x	0	$\frac{\pi}{16}$	$\frac{\pi}{8}$	$\frac{3\pi}{16}$	$\frac{\pi}{4}$
y	1	1.019591	1.082392	1.20269	$\sqrt{2}$

$h=\frac{\pi}{16}$

(b) $I=\int_0^{\frac{\pi}{4}}\sec x\,dx$ $\qquad I\approx\frac{h}{2}\left[y_0+y_n+2(y_1+\dots+y_{n-1})\right]$

$$I\approx\frac{\frac{\pi}{16}}{2}\left[1+\sqrt{2}+2(1.019591+1.082392+1.20269)\right]$$

$$=\underline{0.8859}\quad(4sf)$$

(c) $$\int_0^{\frac{\pi}{4}}\sec x\,dx=\Big[\ln|\tan x+\sec x|\Big]_0^{\frac{\pi}{4}}$$

$$=\left[\ln\left(\tan\tfrac{\pi}{4}+\sec\tfrac{\pi}{4}\right)-\ln\left(\tan 0+\sec 0\right)\right]$$

$$=\ln(1+\sqrt{2})-\ln 1$$

$$=\ln(1+\sqrt{2})$$

$$\%\text{ error}=\frac{0.8859-\ln(1+\sqrt{2})}{\ln(1+\sqrt{2})}\times100\%=0.512\%$$

Q3 $$I=\int_1^5\frac{3x}{\sqrt{(2x-1)}}\,dx$$

$u^2=2x-1$ Differentiating implicitly

so, $u=\sqrt{(2x-1)}$ $\quad 2u\frac{du}{dx}=2$

and by rearranging $\quad u\frac{du}{dx}=1 \Rightarrow \underline{dx=u\,du}$

$\underline{x=\frac{1}{2}(u^2+1)}$

$$I=\int_{u(1)}^{u(5)}\frac{3\times\frac{1}{2}(u^2+1)}{u}\times u\,du$$

$u=\sqrt{(2x-1)}$

limits are changed to:

$u(1)=\sqrt{2\times1-1}=1$ and $u(5)=\sqrt{2\times5-1}=3$

$$=\int_1^3\frac{\frac{3}{2}(u^2+1)}{\cancel{u}}\times\cancel{u}\,du$$

$$=\int_1^3\frac{3}{2}(u^2+1)\,du$$

$$=\left[\frac{3}{2}\left(\frac{1}{3}u^3+u\right)\right]_1^3=\frac{3}{2}\left[\left(\frac{1}{3}\times3^3+3\right)-\left(\frac{1}{3}\times1^3+1\right)\right]$$

$$=\frac{3}{2}\left[12-\frac{4}{3}\right]=\frac{3}{2}\times\frac{32}{3}=16$$

Q4 $$\text{Volume}=\pi\int_1^3y^2\,dx=\pi\int_1^3x^2e^{2x}\,dx$$

$I_1=\int x^2e^{2x}\,dx$ Integrating by parts:

$u=x^2 \qquad \frac{dv}{dx}=e^{2x}$

$\frac{du}{dx}=2x \qquad v=\frac{1}{2}e^{2x}$

$$\Rightarrow I_1=\frac{1}{2}x^2e^{2x}-\int xe^{2x}\,du$$

$I_2=\int xe^{2x}\,dx$ Integrating by parts again:

$u=x \qquad \frac{dv}{dx}=e^{2x}$

$\frac{du}{dx}=1 \qquad v=\frac{1}{2}e^{2x}$

$$\Rightarrow I_2=\frac{1}{2}xe^{2x}-\frac{1}{2}\int e^{2x}\,dx$$

$$=\frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}$$

So, $$I_1=\frac{1}{2}x^2e^{2x}-\left(\frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}\right)$$

$$=\frac{1}{2}x^2e^{2x}-\frac{1}{2}xe^{2x}+\frac{1}{4}e^{2x}$$

Therefore $$\text{Volume}=\pi\left[\frac{1}{2}x^2e^{2x}-\frac{1}{2}xe^{2x}+\frac{1}{4}e^{2x}\right]_1^3$$

$$=\pi\left[\left(\frac{1}{2}\times3^2e^{2\times3}-\frac{1}{2}\times3e^{2\times3}+\frac{1}{4}e^{2\times3}\right)-\left(\frac{1}{2}\times1^2e^2-\frac{1}{2}\times1\times e^2+\frac{1}{4}e^2\right)\right]$$

$$=\pi\left[3.25e^6-0.25e^2\right]$$

$$=\underline{\pi\left(\frac{13}{4}e^6-\frac{1}{4}e^2\right)}$$

Q5 $f(x)=\dfrac{3x^2+16}{(1-3x)(2+x)^2}=\dfrac{A}{(1-3x)}+\dfrac{B}{(2+x)}+\dfrac{C}{(2+x)^2}$

$=\dfrac{A(2+x)^2+B(2+x)(1-3x)+C(1-3x)}{(1-3x)(2+x)^2}$

Comparing the numerators:

$3x^2+16\equiv A(2+x)^2+B(2+x)(1-3x)+C(1-3x)$

let $x=-2$, $3\times(-2)^2+16=C(1-3\times(-2))$

$28=7C$

$\underline{C=4}$

let $x=\frac{1}{3}$, $3\times(\frac{1}{3})^2+16=A(2+\frac{1}{3})^2$

$\frac{49}{3}=\frac{49}{9}A$

$\underline{\underline{A=3}}$

let $x=0$, $16=4A+2B+C$

$16=4\times3+2B+4$

$2B=16-12-4$

$\underline{\underline{B=0}}$

Q5 (b) $f(x)=\dfrac{3}{(1-3x)}+\dfrac{4}{(2+x)^2}$

$=3(1-3x)^{-1}+4(2+x)^{-2}$

$=3(1-3x)^{-1}+4\times2^{-2}(1+\frac{x}{2})^{-2}$

$=3(1-3x)^{-1}+(1+\frac{x}{2})^{-2}$

$3(1-3x)^{-1}=3\left(1+(-1)(-3x)+\frac{(-1)(-2)}{2!}(-3x)^2+\frac{(-1)(-2)(-3)}{3!}(-\right.$

$3\quad =3(1+3x+9x^2+27x^3)$

$=\underline{3+9x+27x^2+81x^3}$

$(1+\frac{x}{2})^{-2}=1+(-2)(\frac{x}{2})+\frac{(-2)(-3)}{2!}(\frac{x}{2})^2+\frac{(-2)(-3)(-4)}{3!}(\frac{x}{2}$

$=\underline{1-x+\frac{3}{4}x^2-\frac{1}{2}x^3}$

$\underline{\underline{f(x)=4+8x+\frac{111}{4}x^2+\frac{161}{2}x^3}}$

Q6 (a) $l_1: \underline{r}=\begin{pmatrix}8+\lambda\\12+\lambda\\14-\lambda\end{pmatrix}$ A: $\begin{pmatrix}4\\8\\a\end{pmatrix}=\begin{pmatrix}8+\lambda_1\\12+\lambda_1\\14-\lambda_1\end{pmatrix}\begin{matrix}(i)\\(ii)\\(iii)\end{matrix}$

Using (i) $4=8+\lambda_1$

$\underline{\lambda_1=-4}$

and (iii)

$a=14-\lambda_1$

$\underline{\underline{a=18}}$

B: $\begin{pmatrix}b\\13\\13\end{pmatrix}=\begin{pmatrix}8+\lambda_2\\12+\lambda_2\\14-\lambda_2\end{pmatrix}\begin{matrix}(i)\\(ii)\\(iii)\end{matrix}$

Using (ii) $\lambda_2=1$

and (i) $\underline{\underline{b=8+1=9}}$

(b) P is a point on l_1 so $\overrightarrow{OP}=\begin{pmatrix}8+\mu\\12+\mu\\14-\mu\end{pmatrix}$

$\overrightarrow{OP}$ is perpendicular to l_1

and therefore to d_1 (the direction vector of l_1)

$\overrightarrow{OP}\cdot\underline{d}_1=0$

$\begin{pmatrix}8+\mu\\12+\mu\\14-\mu\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-1\end{pmatrix}=0$

$(8+\mu)\times1+(12+\mu)\times1+(14-\mu)\times(-1)=0$

$8+\mu+12+\mu-14+\mu=0$

$3\mu=-6$

$\underline{\mu=-2}$

Hence $P(6,10,16)$ ← obtained by substituting $\mu=-2$ into $\begin{pmatrix}8+\mu\\12+\mu\\14-\mu\end{pmatrix}$

Q7 (a) $V=\frac{4}{3}\pi r^3 \Rightarrow \dfrac{dV}{dr}=\frac{4}{3}\pi\times3r^2=\underline{\underline{4\pi r^2}}$

(b) $\dfrac{dr}{dt}=\dfrac{dr}{dV}\times\dfrac{dV}{dt}=\dfrac{1}{\frac{dV}{dr}}\times\dfrac{dV}{dt}$

$=\dfrac{1}{4\pi r^2}\times\dfrac{1000}{(2t+1)^2}=\underline{\dfrac{250}{\pi r^2(2t+1)^2}}$

(c) $\dfrac{dV}{dt}=\dfrac{1000}{(2t+1)^2}$

$\int dV=\int\dfrac{1000\,dt}{(2t+1)^2}$

$V=-500(2t+1)^{-1}+C$

$V=0$ at $t=0$

$0=-500+C$

$\underline{C=500}$

$\underline{V=-500(2t+1)^{-1}+500}$

$\int\dfrac{1000\,dt}{(2t+1)^2}$ Let $u=(2t+1)$

$=\int\dfrac{1000}{u^2}\times\dfrac{du}{2}$ $\quad \dfrac{du}{dt}=2$

$=\int500u^{-2}\,du \Rightarrow [dt=$

$=-500u^{-1}+C$

$=-500(2t+1)^{-1}+C$

(d) at $t=5$ $\quad V(5)=-500(2\times5+1)^{-1}+500$

(i) $=-\frac{500}{11}+500$

$=\frac{5000}{11}$

So $\frac{4}{3}\pi r^3=\frac{5000}{11}$

$r=\sqrt[3]{\frac{3}{4\pi}\times\frac{5000}{11}}=\underline{4.76976}$

Q7 (d)(ii)

$$\frac{dr}{dt} = \frac{250}{\pi r^2 (2t+1)^2} \qquad T = 4.76976 \qquad t = 5$$

$$= 0.0289$$

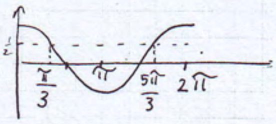
$$= \underline{2.89 \times 10^{-2}}$$

Q8 $x = t - 2\sin t, \quad y = 1 - 2\cos t,$

(a) when $y = 0$ then the curve intercepts the x-axis

$1 - 2\cos t = 0$

$\cos t = \frac{1}{2} \Rightarrow t = \frac{\pi}{3}, \frac{5\pi}{3}$



(b)

$$\text{Area} = I_1 = \int_{x(\frac{\pi}{3})}^{x(\frac{5\pi}{3})} y\,dx = \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} y \frac{dx}{dt}\,dt$$

$$\frac{dx}{dt} = 1 - 2\cos t$$

$$= \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1-2\cos t)(1-2\cos t)\,dt$$

$$= \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1-2\cos t)^2\,dt$$

Q8 (c)

$$\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1-2\cos t)^2\,dt = \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1 - 4\cos t + 4\cos^2 t)\,dt$$

$\cos 2t = \cos^2 t - \sin^2 t$

$\cos 2t = 2\cos^2 t - 1$

$2\cos^2 t = (\cos 2t + 1)$

$$= \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1 - 4\cos t + 2\cos 2t + 2)\,dt = \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (3 - 4\cos t + 2\cos 2t)\,dt$$

$$= \left[3t - 4\sin t + \frac{2}{2}\sin 2t\right]_{\frac{\pi}{3}}^{\frac{5\pi}{3}}$$

$$= \left[3 \times \frac{5\pi}{3} - 4\sin\frac{5\pi}{3} + \sin\left(2 \times \frac{5\pi}{3}\right)\right] - \left[3 \times \frac{\pi}{3} - 4\sin\frac{\pi}{3} + \sin\frac{2\pi}{3}\right]$$

$$= \left(5\pi - 4\left(-\frac{\sqrt{3}}{2}\right) - \frac{\sqrt{3}}{2}\right) - \left(\pi - \frac{4\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right)$$

$$= 4\pi + \frac{3\sqrt{3}}{2} + \frac{3\sqrt{3}}{2}$$

$$= \underline{4\pi + 3\sqrt{3}}$$