

C4 June 2012

1.
$$f(x) = \frac{1}{x(3x-1)^2} = \frac{A}{x} + \frac{B}{(3x-1)} + \frac{C}{(3x-1)^2}$$

(a) Find the values of the constants A , B and C .

(4)

(b) (i) Hence find $\int f(x) dx$.

(ii) Find $\int_1^2 f(x) dx$, leaving your answer in the form $a + \ln b$, where a and b are constants.

(6)

a)
$$1 = A(3x-1)^2 + B(x)(3x-1) + C(x)$$

$x=0 \Rightarrow 1 = A$

$x = \frac{1}{3} \Rightarrow 1 = \frac{1}{3}C \Rightarrow C = 3$

$x = 1 \Rightarrow 1 = 4A + 2B + C \Rightarrow 1 = 4 + 2B + 3 \quad B = -3$

b)
$$\int \frac{1}{x} + \frac{-3}{3x-1} + \frac{3}{(3x-1)^2} dx$$

$(3x-1)^{-1}$
 \downarrow

$-1(3x-1)^{-2} \times 3$

$$\therefore \ln x - \ln(3x-1) - \frac{1}{3x-1} + C$$

(ii)
$$\left[\ln \left(\frac{x}{3x-1} \right) - \frac{1}{3x-1} \right]_1^2$$

$$= \ln \frac{2}{5} - \frac{1}{5} - \ln \frac{1}{2} + \frac{1}{2} = \frac{3}{10} + \ln \frac{4}{5}$$

2.

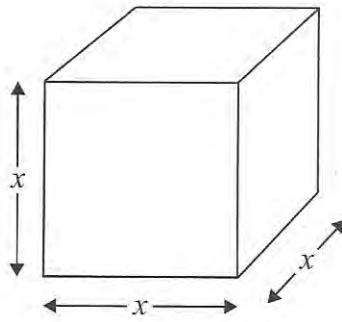


Figure 1

Figure 1 shows a metal cube which is expanding uniformly as it is heated. At time t seconds, the length of each edge of the cube is x cm, and the volume of the cube is V cm³.

(a) Show that $\frac{dV}{dx} = 3x^2$ (1)

Given that the volume, V cm³, increases at a constant rate of 0.048 cm³s⁻¹,

(b) find $\frac{dx}{dt}$, when $x = 8$ (2)

(c) find the rate of increase of the total surface area of the cube, in cm²s⁻¹, when $x = 8$ (3)

$$a) V = x^3 \Rightarrow \frac{dV}{dx} = 3x^2 \#$$

$$b) \frac{dV}{dt} = 0.048 \quad \frac{dx}{dV} \times \frac{dV}{dt} = \frac{0.048}{3x^2}$$

$$x = 8 \quad \frac{dx}{dt} = \frac{1}{4000}$$

$$c) A = 6x^2 \Rightarrow \frac{dA}{dx} = 12x$$

$$\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt} = \frac{12x}{4000} = \frac{3}{125}$$

3.

$$f(x) = \frac{6}{\sqrt{(9-4x)}}, \quad |x| < \frac{9}{4}$$

- (a) Find the binomial expansion of $f(x)$ in ascending powers of x , up to and including the term in x^3 . Give each coefficient in its simplest form.

(6)

Use your answer to part (a) to find the binomial expansion in ascending powers of x , up to and including the term in x^3 , of

(b) $g(x) = \frac{6}{\sqrt{(9+4x)}}, \quad |x| < \frac{9}{4}$ (1)

(c) $h(x) = \frac{6}{\sqrt{(9-8x)}}, \quad |x| < \frac{9}{8}$ (2)

$$f(x) = 6(9-4x)^{-\frac{1}{2}} = 6 \times 9^{-\frac{1}{2}} \left(1 - \frac{4}{9}x\right)^{-\frac{1}{2}} = 2\left(1 - \frac{4}{9}x\right)^{-\frac{1}{2}}$$

$$= 2 \left[1 + \left(-\frac{1}{2}\right)\left(-\frac{4}{9}x\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2} \left(-\frac{4}{9}x\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{6} \left(-\frac{4}{9}x\right)^3 \right]$$

$$= 2 + \frac{4}{9}x + \frac{4}{27}x^2 + \frac{40}{729}x^3$$

$$b) g(x) = 2 - \frac{4}{9}x + \frac{4}{27}x^2 - \frac{40}{729}x^3$$

$$c) h(x) = 2 + \frac{8}{9}x + \frac{16}{27}x^2 + \frac{320}{729}x^3$$

4. Given that $y = 2$ at $x = \frac{\pi}{4}$, solve the differential equation

$$\frac{dy}{dx} = \frac{3}{y \cos^2 x}$$

(5)

$$\int y dy = \int 3 \sec^2 x dx$$

$$\frac{1}{2} y^2 = 3 \tan x + C$$

$$\left(\frac{\pi}{4}, 2\right) \quad 2 = 3 \tan \frac{\pi}{4} + C \quad \Rightarrow \quad 2 = 3 + C \quad \Rightarrow \quad C = -1$$

$$\therefore \frac{1}{2} y^2 = 3 \tan x - 1$$

$$y = \sqrt{6 \tan x - 2}$$

5. The curve C has equation

$$16y^3 + 9x^2y - 54x = 0$$

(a) Find $\frac{dy}{dx}$ in terms of x and y .

(5)

(b) Find the coordinates of the points on C where $\frac{dy}{dx} = 0$.

(7)

$$\frac{d}{dx} 16y^3 + \frac{d}{dx} 9x^2y - \frac{d}{dx} 54x = 0$$

$$\frac{dy}{dx} 48y^2 + 9x^2 \frac{dy}{dx} + 18xy - 54 = 0$$

$$u = 9x^2 \quad v = y \\ u' = 18x \quad v' = \frac{dy}{dx}$$

$$(48y^2 + 9x^2) \frac{dy}{dx} = 54 - 18xy$$

$$\therefore \frac{dy}{dx} = \frac{54 - 18xy}{48y^2 + 9x^2} = \frac{18 - 6xy}{16y^2 + 3x^2}$$

$$b) \frac{dy}{dx} = 0 \Rightarrow 6xy = 18 \Rightarrow xy = 3 \Rightarrow y = \frac{3}{x}$$

$$16\left(\frac{3}{x}\right)^3 + 9x^2\left(\frac{3}{x}\right) - 54x = 0$$

$$\frac{432}{x^3} + 27x - 54x = 0$$

$$\Rightarrow \frac{432}{x^3} = 27x \Rightarrow x^4 = \frac{432}{27} = 16$$

$$\therefore x = \pm 2 \quad \left(2, \frac{3}{2}\right); \left(-2, -\frac{3}{2}\right)$$

6.

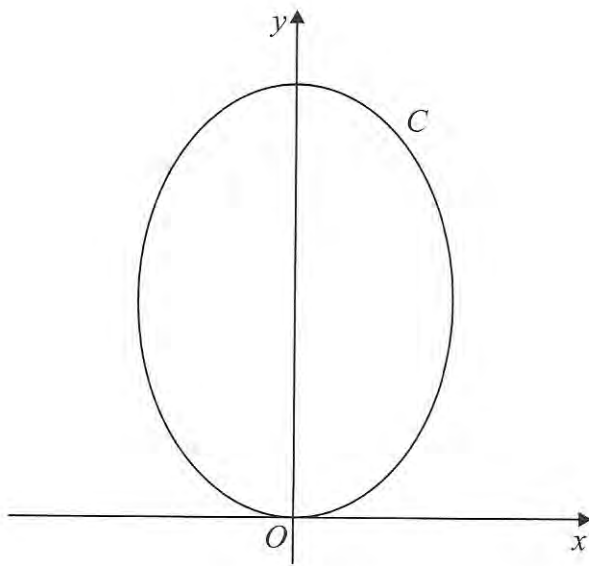


Figure 2

Figure 2 shows a sketch of the curve C with parametric equations

$$x = (\sqrt{3})\sin 2t, \quad y = 4 \cos^2 t, \quad 0 \leq t \leq \pi$$

(a) Show that $\frac{dy}{dx} = k(\sqrt{3})\tan 2t$, where k is a constant to be determined. (5)

(b) Find an equation of the tangent to C at the point where $t = \frac{\pi}{3}$.

Give your answer in the form $y = ax + b$, where a and b are constants. (4)

(c) Find a cartesian equation of C . (3)

$$\frac{dy}{dt} = -8 \cos t \sin t \Rightarrow \frac{dy}{dx} = \frac{-4 \sin 2t}{2\sqrt{3} \cos 2t}$$

$$\frac{dx}{dt} = 2\sqrt{3} \cos 2t$$

$$\frac{dy}{dx} = -\frac{2\sqrt{3}}{3} \tan 2t \quad (k = -\frac{2}{3})$$

$$b) \quad t = \frac{\pi}{3} \quad x = \sqrt{3} \left(\frac{\sqrt{3}}{2} \right) = \frac{3}{2} \quad y = 4 \left(\frac{1}{2} \right)^2 = 1$$

$$m_t = -\frac{2\sqrt{3}}{3} (-\sqrt{3}) = 2 \Rightarrow y - 1 = 2 \left(x - \frac{3}{2} \right)$$

$$y = 2x - 2$$

$$c) x = \sqrt{3} \times 2 \sin t \cos t$$

$$\frac{y}{4} = \cos^2 t$$

$$x^2 = 12 \sin^2 t \cos^2 t$$

$$\frac{y}{4} = 1 - \sin^2 t$$

$$\therefore x^2 = 12 \left(1 - \frac{y}{4}\right) \left(\frac{y}{4}\right)$$

$$\sin^2 t = 1 - \frac{y}{4}$$

the following is not required.

$$x^2 = 12 \left(\frac{y}{4} - \frac{y^2}{16}\right) \Rightarrow x^2 = 3y - \frac{3}{4}y^2$$

$$\therefore 3y^2 - 12y + 4x^2 = 0$$

$$y^2 - 4y + \frac{4}{3}x^2 = 0$$

$$(y-2)^2 = 4 - \frac{4}{3}x^2$$

$$y-2 = \pm \sqrt{4 - \frac{4}{3}x^2}$$

$$\therefore y = 2 \pm \sqrt{4 - \frac{4}{3}x^2}$$

alt

$$\sin 2t = \frac{x}{\sqrt{3}} \Rightarrow 2t = \arcsin\left(\frac{x}{\sqrt{3}}\right)$$

$$t = \frac{1}{2} \arcsin\left(\frac{x}{\sqrt{3}}\right)$$

$$\therefore y = 4 \cos^2 \left[\frac{1}{2} \arcsin\left(\frac{x}{\sqrt{3}}\right) \right]$$

7.

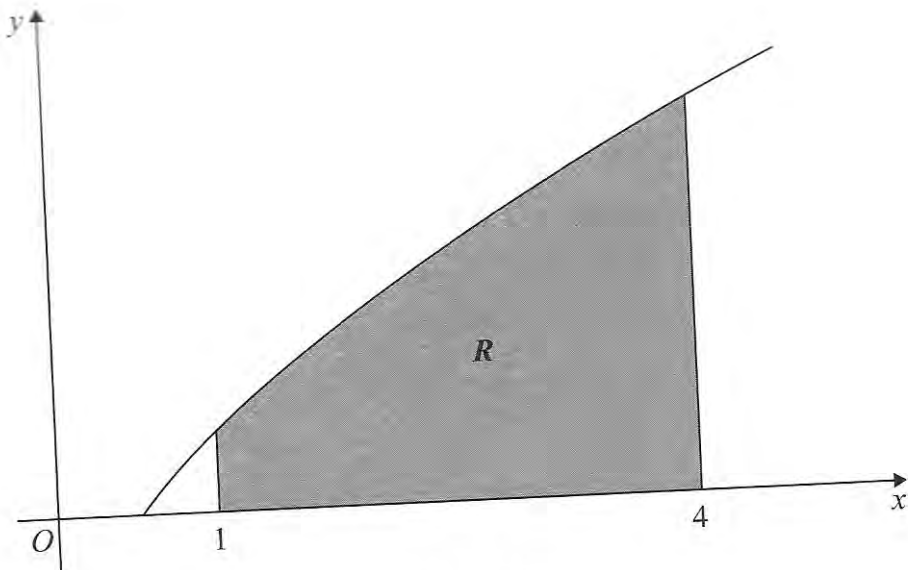


Figure 3

Figure 3 shows a sketch of part of the curve with equation $y = x^{\frac{1}{2}} \ln 2x$.

The finite region R , shown shaded in Figure 3, is bounded by the curve, the x -axis and the lines $x = 1$ and $x = 4$.

- (a) Use the trapezium rule, with 3 strips of equal width, to find an estimate for the area of R , giving your answer to 2 decimal places. (4)
- (b) Find $\int x^{\frac{1}{2}} \ln 2x \, dx$. (4)
- (c) Hence find the exact area of R , giving your answer in the form $a \ln 2 + b$, where a and b are exact constants. (3)

x	1	2	3	4
y	$\ln 2$	$\sqrt{2} \ln 4$	$\sqrt{3} \ln 6$	$2 \ln 8$

$$\text{Area} \approx \frac{1}{2}(1) \left[\ln 2 + 2 \ln 8 + 2(\sqrt{2} \ln 4 + \sqrt{3} \ln 6) \right]$$

$$\text{Area} \approx \frac{14.98}{2} = 7.49$$

$$b) \int x^{\frac{1}{2}} \ln 2x \, dx \quad \int uv' = uv - \int u'v$$

$$u = \ln 2x \quad v = \frac{2}{3}x^{\frac{3}{2}}$$

$$u' = \frac{1}{x} \quad v' = x^{\frac{1}{2}}$$

$$= \frac{2}{3}x^{\frac{3}{2}} \ln 2x - \int \frac{\frac{2}{3}x^{\frac{3}{2}}}{x} \, dx$$

$$= \frac{2}{3}x^{\frac{3}{2}} \ln 2x - \frac{4}{9}x^{\frac{3}{2}} + C$$

$$c) \int_1^4 x^{\frac{1}{2}} \ln 2x \, dx = \left[\frac{2}{3}x^{\frac{3}{2}} \ln 2x - \frac{4}{9}x^{\frac{3}{2}} \right]_1^4$$

$$= \frac{16}{3} \ln 8 - \frac{32}{9} - \frac{2}{3} \ln 2 + \frac{4}{9}$$

$$= 16 \ln 2 - \frac{2}{3} \ln 2 - \frac{28}{9}$$

$$= \frac{46}{3} \ln 2 - \frac{28}{9}$$

8. Relative to a fixed origin O , the point A has position vector $(10\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$, and the point B has position vector $(8\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})$.

The line l passes through the points A and B .

(a) Find the vector \overrightarrow{AB} .

(2)

(b) Find a vector equation for the line l .

(2)

The point C has position vector $(3\mathbf{i} + 12\mathbf{j} + 3\mathbf{k})$.

The point P lies on l . Given that the vector \overrightarrow{CP} is perpendicular to l ,

(c) find the position vector of the point P .

(6)

$$a = \begin{pmatrix} 10 \\ 2 \\ 3 \end{pmatrix} \quad b = \begin{pmatrix} 8 \\ 3 \\ 4 \end{pmatrix} \quad \overrightarrow{AB} = b - a = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

$$b) \quad \begin{pmatrix} 10 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 - 2\lambda \\ 2 + \lambda \\ 3 + \lambda \end{pmatrix}$$

$$c) \quad c = \begin{pmatrix} 3 \\ 12 \\ 3 \end{pmatrix}$$

$$\overrightarrow{CP} = \begin{pmatrix} 10 - 2\lambda \\ 2 + \lambda \\ 3 + \lambda \end{pmatrix} - \begin{pmatrix} 3 \\ 12 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 - 2\lambda \\ -10 + \lambda \\ \lambda \end{pmatrix}$$

$$\overrightarrow{CP} \text{ perp to } L \Rightarrow \begin{pmatrix} 7 - 2\lambda \\ -10 + \lambda \\ \lambda \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = 0$$

$$-14 + 4\lambda - 10 + \lambda + \lambda = 0 \Rightarrow 6\lambda = 24 \Rightarrow \lambda = 4$$

$$\therefore p = \begin{pmatrix} 10 - 8 \\ 2 + 4 \\ 3 + 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ 7 \end{pmatrix}$$