

C4 JUNE 2010

1.

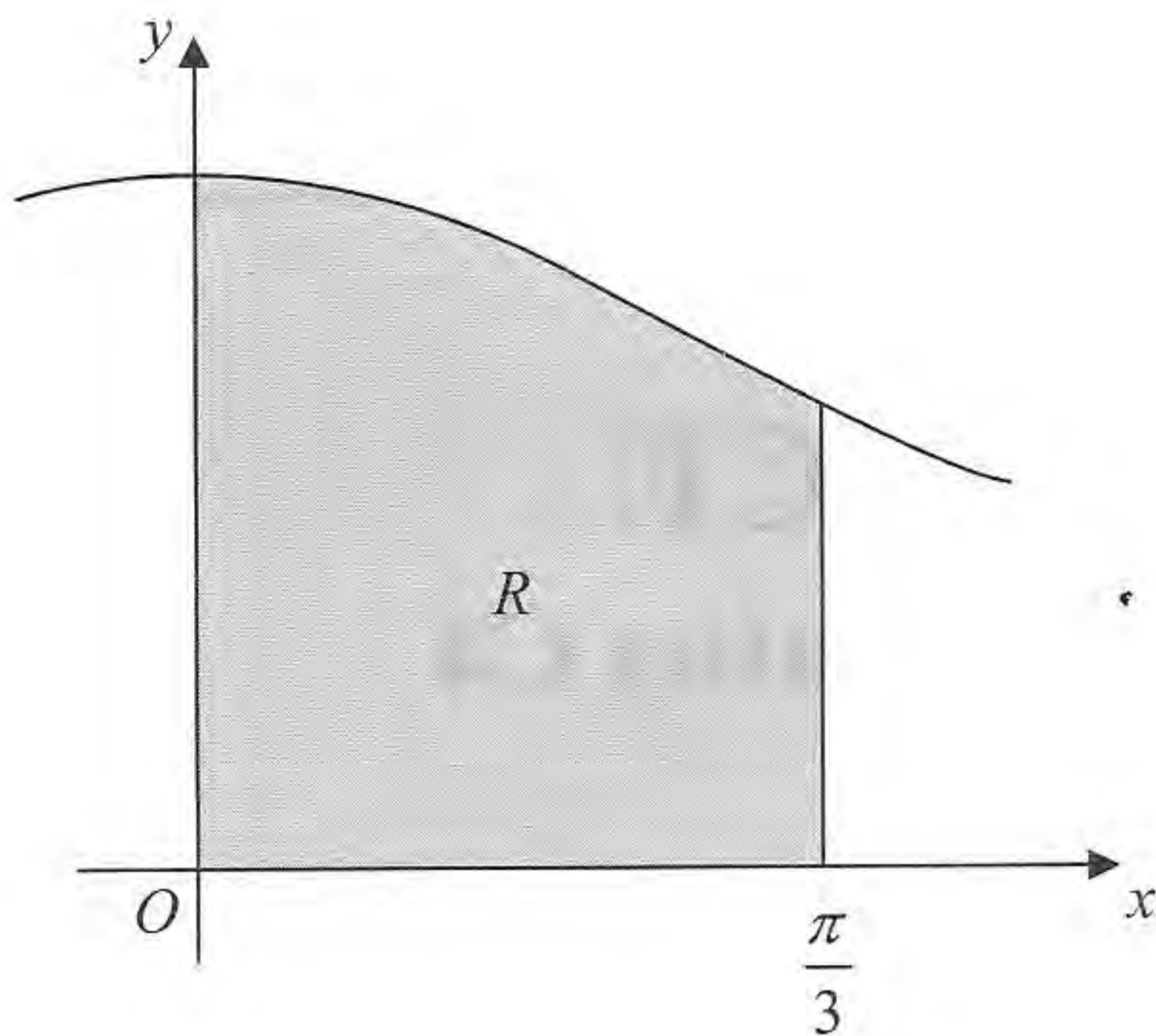


Figure 1

Figure 1 shows part of the curve with equation $y = \sqrt{(0.75 + \cos^2 x)}$. The finite region R , shown shaded in Figure 1, is bounded by the curve, the y -axis, the x -axis and the line with equation $x = \frac{\pi}{3}$.

(a) Complete the table with values of y corresponding to $x = \frac{\pi}{6}$ and $x = \frac{\pi}{4}$.

x	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
y	1.3229	1.2973	1.2247	1.1180	1

(2)

(b) Use the trapezium rule

(i) with the values of y at $x = 0$, $x = \frac{\pi}{6}$ and $x = \frac{\pi}{3}$ to find an estimate of the area of R .
Give your answer to 3 decimal places.

(ii) with the values of y at $x = 0$, $x = \frac{\pi}{12}$, $x = \frac{\pi}{6}$, $x = \frac{\pi}{4}$ and $x = \frac{\pi}{3}$ to find a further estimate of the area of R . Give your answer to 3 decimal places.

$$\text{i) } \frac{1}{2} \left(\frac{\pi}{6} \right) (1.3229 + 2(1.2247) + 1) = \underline{1.249} \text{ (3dp)}^{(6)}$$

$$\text{ii) } \frac{1}{2} \left(\frac{\pi}{12} \right) (1.3229 + 2(1.2973 + 1.2247 + 1.118) + 1) = \underline{1.257} \text{ (3dp)}$$

2. Using the substitution $u = \cos x + 1$, or otherwise, show that

$$\int_0^{\frac{\pi}{2}} e^{\cos x + 1} \sin x \, dx = e(e-1)$$

(6)

$$u = \cos x + 1 \Rightarrow \frac{du}{dx} = -\sin x \Rightarrow dx = \frac{-1}{\sin x} du$$

$$x = \frac{\pi}{2} \quad u = \cos\left(\frac{\pi}{2}\right) + 1 = 1$$

$$x = 0 \quad u = \cos(0) + 1 = 2$$

$$\Rightarrow \int_2^1 e^u \cancel{\sin x} \times \frac{-1}{\cancel{\sin x}} du = -\int_2^1 e^u du = +\int_1^2 e^u du$$

$$= [e^u]_1^2 = e^2 - e^1 = (e^1)^2 - e^1 = e^1(e^1 - 1)$$

$$= \underline{e(e-1)} \text{ qed}$$

3. A curve C has equation

$$2^x + y^2 = 2xy$$

Find the exact value of $\frac{dy}{dx}$ at the point on C with coordinates $(3, 2)$.

(7)

$$\frac{d}{dx} 2^x + \frac{d}{dx} y^2 = \frac{d}{dx} 2xy$$

$$\frac{d}{dx} 2^x \quad u = 2^x \Rightarrow \ln u = \ln 2^x \Rightarrow \ln u = x \ln 2$$

$$\frac{d}{dx} \ln u = \frac{d}{dx} x \ln 2 \Rightarrow \frac{1}{u} \frac{du}{dx} = \ln 2$$

$$\frac{du}{dx} = u \ln 2 \Rightarrow \frac{du}{dx} = 2^x \ln 2$$

$$\frac{d}{dx} y^2 = 2y \frac{dy}{dx} \quad \frac{d}{dx} 2xy \quad \begin{array}{l} u = 2x \\ u' = 2 \end{array} \quad \begin{array}{l} v = y \\ \frac{dv}{dx} = \frac{dy}{dx} \end{array}$$

$$\frac{d}{dx} 2xy = 2y + 2x \frac{dy}{dx}$$

$$\Rightarrow 2^x \ln 2 + 2y \frac{dy}{dx} = 2y + 2x \frac{dy}{dx}$$

$$-2y + 2^x \ln 2 = (2x - 2y) \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2y + 2^x \ln 2}{2x - 2y}$$

$$(3, 2) \Rightarrow \frac{dy}{dx} = \frac{-4 + 8 \ln 2}{6 - 4} = \frac{-4 + 8 \ln 2}{2} = \underline{\underline{-2 + 4 \ln 2}}$$

4. A curve C has parametric equations

$$x = \sin^2 t, \quad y = 2 \tan t, \quad 0 \leq t < \frac{\pi}{2}$$

(a) Find $\frac{dy}{dx}$ in terms of t .

(4)

The tangent to C at the point where $t = \frac{\pi}{3}$ cuts the x -axis at the point P .

(b) Find the x -coordinate of P .

(6)

$$\frac{dx}{dt} = 2 \sin t \cos t \quad \frac{dy}{dt} = 2 \sec^2 t = \frac{2}{\cos^2 t}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{2}{\cos^2 t} \times \frac{1}{2 \sin t \cos t} = \frac{1}{\sin t \cos^3 t}$$

$$\text{b) } t = \frac{\pi}{3} \quad x = \sin^2\left(\frac{\pi}{3}\right) = \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4}$$

$$y = 2 \tan\left(\frac{\pi}{3}\right) = 2\sqrt{3}$$

$$m_t \text{ at } t = \frac{\pi}{3} = \frac{1}{\sin\left(\frac{\pi}{3}\right) \cos^3\left(\frac{\pi}{3}\right)} = \frac{1}{\left(\frac{\sqrt{3}}{2}\right) \left(\frac{1}{2}\right)^3} = \frac{16}{\sqrt{3}}$$

$$y - 2\sqrt{3} = \frac{16}{\sqrt{3}} \left(x - \frac{3}{4}\right) \Rightarrow \sqrt{3}y - 6 = 16x - 12$$

$$y\sqrt{3} = 16x - 6$$

$$\text{Cuts } x\text{-axis when } y=0 \Rightarrow 16x = 6 \quad x = \frac{6}{16} = \frac{3}{8}$$

5.

$$\frac{2x^2 + 5x - 10}{(x-1)(x+2)} \equiv A + \frac{B}{x-1} + \frac{C}{x+2}$$

(a) Find the values of the constants A , B and C .

(4)

(b) Hence, or otherwise, expand $\frac{2x^2 + 5x - 10}{(x-1)(x+2)}$ in ascending powers of x , as far as theterm in x^2 . Give each coefficient as a simplified fraction.

(7)

$$2x^2 + 5x - 10 = A(x-1)(x+2) + B(x+2) + C(x-1)$$

$$x=1 \Rightarrow -3 = 3B \quad \underline{B=-1}$$

$$x=-2 \Rightarrow -12 = -3C \quad \underline{C=4}$$

$$x=0 \Rightarrow -10 = -2A + 2B - C \Rightarrow -4 = -2A \Rightarrow \underline{A=2}$$

$$b) \frac{2x^2 + 5x - 10}{(x-1)(x+2)} = 2 + \frac{-1}{(x-1)} + \frac{4}{x+2} = 2 + \frac{1}{1-x} + \frac{4}{2+x}$$

$$= 2 + (1-x)^{-1} + 4(2+x)^{-1} = 2 + (1-x)^{-1} + 4 \times 2^{-1} \left(1 + \frac{x}{2}\right)^{-1}$$

$$= 2 + (1-x)^{-1} + 2 \left(1 + \frac{x}{2}\right)^{-1}$$

$$= 2 + \left(1 + (-1)(-x) + \frac{(-1)(-2)(-x)^2}{2}\right) + 2 \left(1 + (-1)\left(\frac{x}{2}\right) + \frac{(-1)(-2)\left(\frac{x}{2}\right)^2}{2}\right)$$

$$= 2 + \left(1 + x + x^2\right) + \left(2 + x + \frac{1}{2}x^2\right)$$

$$= \underline{\underline{5 + \frac{3}{2}x^2}}$$

6.

$$f(\theta) = 4 \cos^2 \theta - 3 \sin^2 \theta$$

(a) Show that $f(\theta) = \frac{1}{2} + \frac{7}{2} \cos 2\theta$. (3)

(b) Hence, using calculus, find the exact value of $\int_0^{\frac{\pi}{2}} \theta f(\theta) d\theta$. (7)

$$\cos 2\theta = 2\cos^2 \theta - 1 \Rightarrow 2\cos^2 \theta = \cos 2\theta + 1 \Rightarrow 4\cos^2 \theta = 2\cos 2\theta + 2$$

$$\cos 2\theta = 1 - 2\sin^2 \theta \Rightarrow 2\sin^2 \theta = 1 - \cos 2\theta \Rightarrow 3\sin^2 \theta = \frac{3}{2} - \frac{3}{2}\cos 2\theta$$

$$\Rightarrow 4\cos^2 \theta - 3\sin^2 \theta = 2\cos 2\theta + 2 - \frac{3}{2} + \frac{3}{2}\cos 2\theta$$

$$= \frac{1}{2} + \frac{7}{2}\cos 2\theta \quad \text{qed.}$$

b) $\int_0^{\frac{\pi}{2}} \theta f(\theta) d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{2}\theta d\theta + \int_0^{\frac{\pi}{2}} \frac{7}{2}\theta \cos 2\theta d\theta$ $u = \frac{7}{2}\theta$ $v = \frac{1}{2}\sin 2\theta$
 $u' = \frac{7}{2}$ $v' = \cos 2\theta$

$$= \left[\frac{1}{4}\theta^2 + \frac{7}{4}\theta \sin 2\theta - \int \frac{7}{4}\sin 2\theta d\theta \right]_0^{\frac{\pi}{2}}$$

$$= \left[\frac{1}{4}\theta^2 + \frac{7}{4}\theta \sin 2\theta + \frac{7}{8}\cos 2\theta \right]_0^{\frac{\pi}{2}}$$

$$= \left(\frac{1}{4}\left(\frac{\pi}{2}\right)^2 + 0 - \frac{7}{8} \right) - \left(0 + 0 + \frac{7}{8} \right) = \frac{\pi^2}{16} - \frac{7}{4} = \frac{1}{16}(\pi^2 - 28)$$

7. The line l_1 has equation $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, where λ is a scalar parameter.

The line l_2 has equation $\mathbf{r} = \begin{pmatrix} 0 \\ 9 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix}$, where μ is a scalar parameter.

Given that l_1 and l_2 meet at the point C , find

(a) the coordinates of C .

(3)

The point A is the point on l_1 where $\lambda = 0$ and the point B is the point on l_2 where $\mu = -1$.

(b) Find the size of the angle ACB . Give your answer in degrees to 2 decimal places.

(4)

(c) Hence, or otherwise, find the area of the triangle ABC .

(5)

$$l_1 = l_2 \Rightarrow \begin{pmatrix} 2 + \lambda \\ 3 + 2\lambda \\ -4 + \lambda \end{pmatrix} = \begin{pmatrix} 5\mu \\ 9 \\ -3 + 2\mu \end{pmatrix} \quad \begin{array}{l} j \Rightarrow 2\lambda = 6 \Rightarrow \lambda = 3 \\ i) \quad 5 = 5\mu \Rightarrow \mu = 1 \end{array}$$

$$\lambda = 3 \Rightarrow C(5, 9, -1)$$

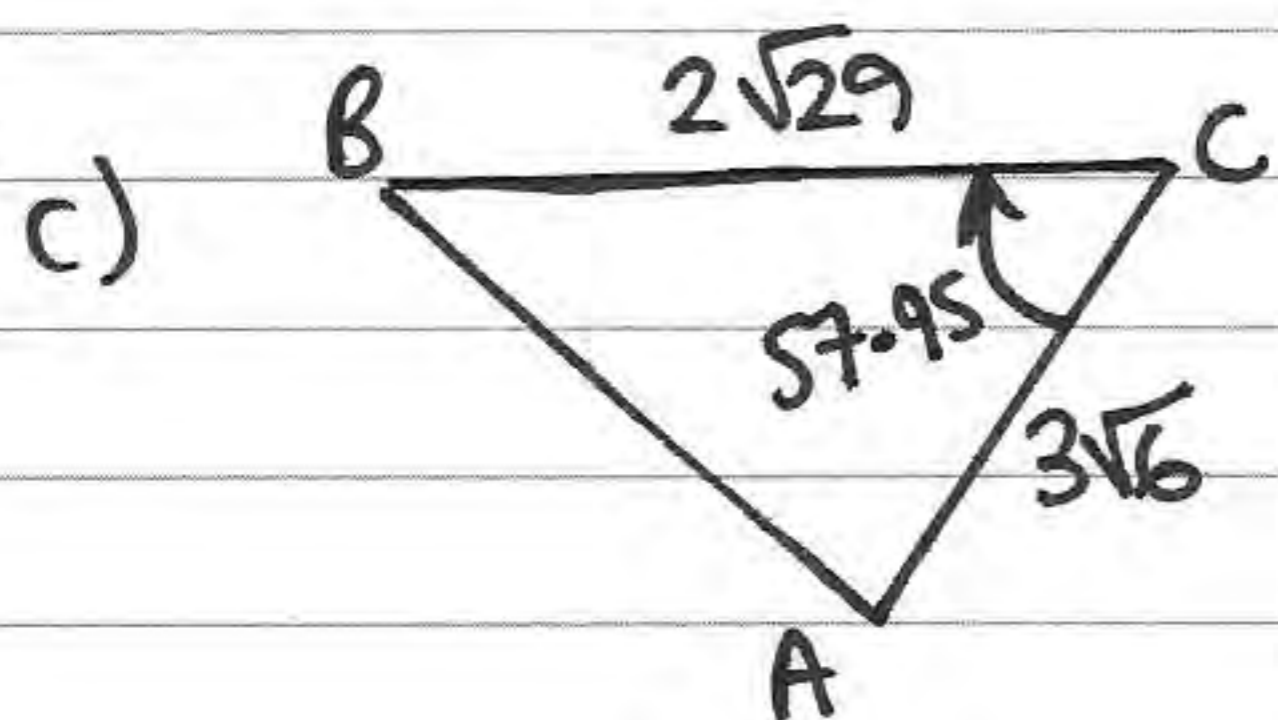
$$b) \quad \lambda = 0 \Rightarrow A(2, 3, -4) \quad \mu = -1 \Rightarrow B(-5, 9, -5)$$

$$\vec{CA} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} - \begin{pmatrix} 5 \\ 9 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ -6 \\ -3 \end{pmatrix} \quad |\vec{CA}| = \sqrt{3^2 + 6^2 + 3^2} = 3\sqrt{6}$$

$$\vec{CB} = \begin{pmatrix} -5 \\ 9 \\ -5 \end{pmatrix} - \begin{pmatrix} 5 \\ 9 \\ -1 \end{pmatrix} = \begin{pmatrix} -10 \\ 0 \\ -4 \end{pmatrix} \quad |\vec{CB}| = \sqrt{10^2 + 4^2} = 2\sqrt{29}$$

$$\vec{CA} \cdot \vec{CB} = 30 + 0 + 12 = 42$$

$$\cos \theta = \frac{\vec{CA} \cdot \vec{CB}}{|\vec{CA}| |\vec{CB}|} = \frac{42}{6\sqrt{174}} \Rightarrow \theta = 57.95^\circ$$



$$\text{Area} = \frac{1}{2} (2\sqrt{29})(3\sqrt{6}) \sin 57.95^\circ \dots$$

$$\text{Area} = \underline{\underline{33.54}}$$

8.

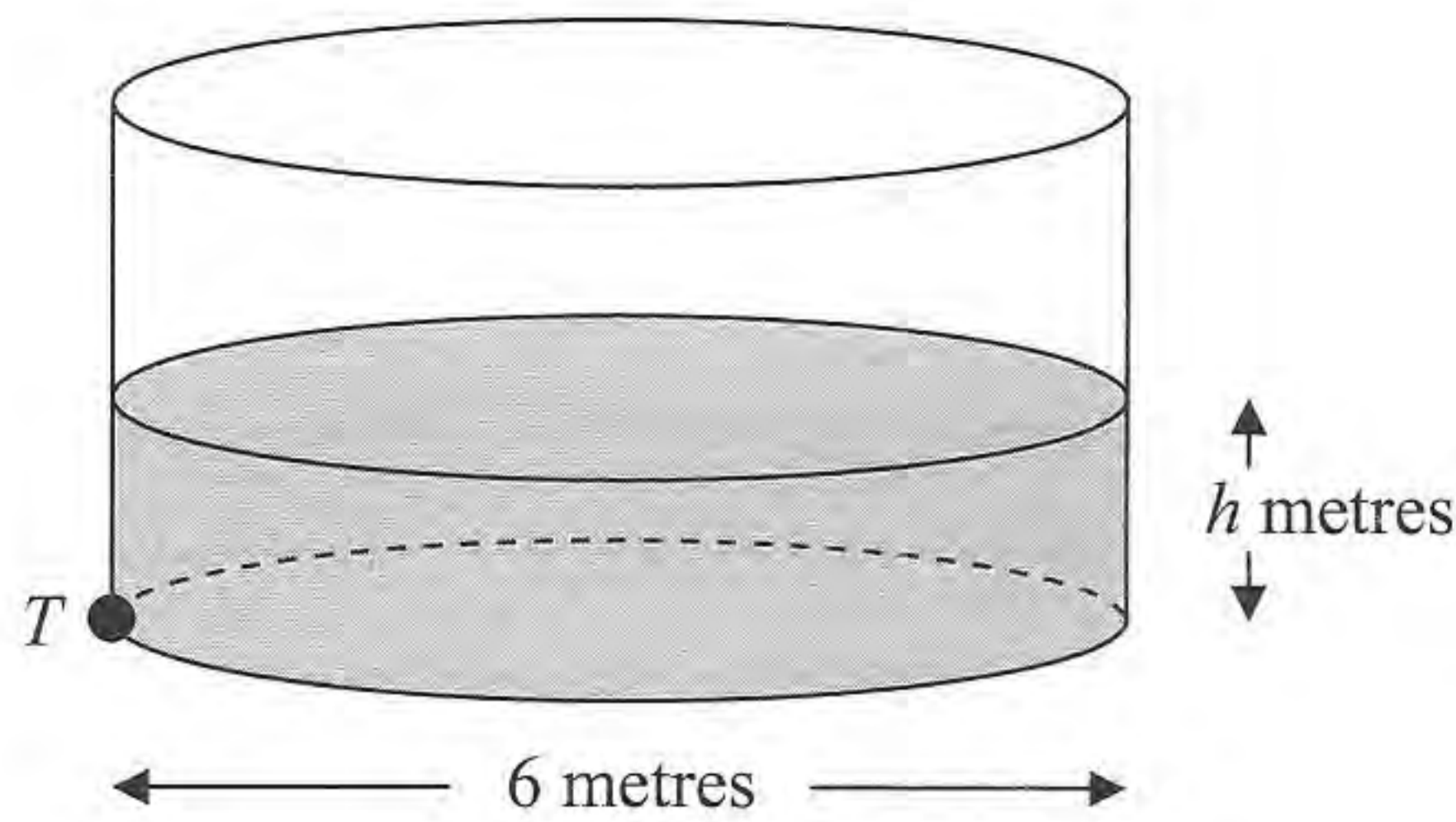


Figure 2

Figure 2 shows a cylindrical water tank. The diameter of a circular cross-section of the tank is 6 m. Water is flowing into the tank at a constant rate of $0.48\pi \text{ m}^3 \text{ min}^{-1}$. At time t minutes, the depth of the water in the tank is h metres. There is a tap at a point T at the bottom of the tank. When the tap is open, water leaves the tank at a rate of $0.6\pi h \text{ m}^3 \text{ min}^{-1}$.

(a) Show that t minutes after the tap has been opened

$$75 \frac{dh}{dt} = (4 - 5h) \quad (5)$$

When $t = 0$, $h = 0.2$

(b) Find the value of t when $h = 0.5$

(6)

$$\text{a) } \frac{dV}{dt} = 0.48\pi - 0.6\pi h \quad V = \pi r^2 h = 9\pi h$$

$$\frac{dV}{dh} = 9\pi \Rightarrow \frac{dh}{dV} = \frac{1}{9\pi}$$

$$\frac{dh}{dt} = \frac{dh}{dV} \frac{dV}{dt} = \frac{1}{9\pi} (0.48\pi - 0.6\pi h)$$

$$\frac{dh}{dt} = \frac{4}{75} - \frac{1}{15}h \Rightarrow \frac{dh}{dt} = \frac{1}{75} (4 - 5h) \Rightarrow 75 \frac{dh}{dt} = 4 - 5h \text{ reqd.}$$

$$\text{b) } -15 \int \frac{-5}{4-5h} dh = \int dt \Rightarrow -15 \ln(4-5h) = t + C$$

$$t=0, h=0.2 \Rightarrow -15 \ln 3 = C$$

$$t = 15 \ln 3 - 15 \ln(4-5h) \Rightarrow t = 15 \ln \left(\frac{3}{4-5h} \right) \quad t = 10.4 \text{ (3sf)}$$

$$h=0.5 \Rightarrow t = 15 \ln \frac{3}{1.5} = 15 \ln 2$$