JAN 12

1. The curve C has the equation $2x + 3y^2 + 3x^2y = 4x^2$.
The point P on the curve has coordinates $(-1, 1)$.

(a) Find the gradient of the curve at P.

(5)

(b) Hence find the equation of the normal to C at P, giving your answer in the form $ax+by+c=0$, where a, b and c are integers.

(3)

a) $\frac{d}{dx}(2x) + \frac{d}{dx}(3y^2) + \frac{d}{dx}(3x^2y) = \frac{d}{dx}(4x^2)$

$\Rightarrow 2 + 6y\frac{dy}{dx} + 6xy + 3x^2\frac{dy}{dx} = 8x$

$\Rightarrow (6y+3x^2)\frac{dy}{dx} = 8x - 6xy - 2$

$\therefore \frac{dy}{dx} = \frac{8x-6xy-2}{6y+3x^2}$

at $(-1,1)$ $M_t = \frac{-8+6-2}{6+3} = \frac{-4}{9}$

b) $M_n = \frac{9}{4}$ $\Rightarrow$ $y-1 = \frac{9}{4}(x+1)$
$(-1,1)$ $\therefore 9x-4y+13=0$

2. (a) Use integration by parts to find $\int x \sin 3x \, dx$.

(3)

(b) Using your answer to part (a), find $\int x^2 \cos 3x \, dx$.

(3)

$\int x \sin 3x \, dx = -\frac{1}{3} x \cos 3x - \int -\frac{1}{3} \cos 3x \, dx$ $\quad u = x \quad v = -\frac{1}{3}\cos 3x$

$u' = 1 \quad v' = \sin 3x$

$= -\frac{1}{3} x \cos 3x + \frac{1}{9} \sin 3x + c$

b) $\int x^2 \cos 3x \, dx = \frac{1}{3} x^2 \sin 3x - \frac{2}{3} \int x \sin 3x \, dx$ $\quad u = x^2 \quad v = \frac{1}{3} \sin 3x$

$u' = 2x \quad v' = \cos 3x$

$= \frac{1}{3} x^2 \sin 3x - \frac{2}{3}\left[-\frac{1}{3} \cos 3x + \frac{1}{9} \sin 3x\right] + c$

$= \frac{1}{3} x^2 \sin 3x + \frac{2}{9} x \cos 3x - \frac{2}{27} \sin 3x + c$

3. (a) Expand

$$\frac{1}{(2-5x)^2}, \quad |x| < \frac{2}{5}$$

in ascending powers of x, up to and including the term in x^2, giving each term as a simplified fraction.

(5)

Given that the binomial expansion of $\dfrac{2+kx}{(2-5x)^2}$, $|x| < \dfrac{2}{5}$, is

$$\frac{1}{2} + \frac{7}{4}x + Ax^2 + \dots$$

(b) find the value of the constant k,

(2)

(c) find the value of the constant A.

(2)

a) $(2-5x)^{-2} = 2^{-2}\left(1-\frac{5}{2}x\right)^{-2} = \frac{1}{4}\left(1-\frac{5}{2}x\right)^{-2}$

$= \frac{1}{4}\left(1 - 2\left(-\frac{5}{2}x\right) + \frac{(-2)(-3)}{2}\left(-\frac{5}{2}x\right)^2\right)$

$= \frac{1}{4} + \frac{5}{4}x + \frac{75}{16}x^2$

b) $\dfrac{2+kx}{(2-5x)^2} = (2+kx)\left(\frac{1}{4} + \frac{5}{4}x + \frac{75}{16}x^2 + \dots\right)$

$\times 2$: $\quad \frac{2}{4} + \frac{10}{4}x + \frac{150}{16}x^2$

$\times kx$: $\quad \frac{1}{4}kx + \frac{5}{4}kx^2 + \dots$

$= \frac{1}{2} + \left(\frac{5}{2} + \frac{1}{4}k\right)x + \left(\frac{75}{8} + \frac{5}{4}k\right)x^2$

$\therefore \frac{5}{2} + \frac{1}{4}k = \frac{7}{4} \Rightarrow \frac{1}{4}k = -\frac{3}{4} \Rightarrow k = -3$

$\therefore A = \frac{75}{8} + \frac{5}{4}(-3) = \frac{45}{8}$

4.

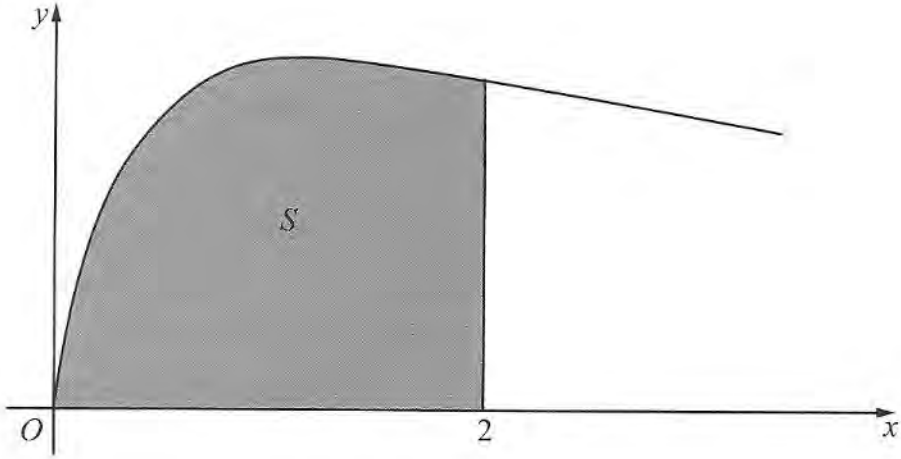

Figure 1

Figure 1 shows the curve with equation

$$y = \sqrt{\left(\frac{2x}{3x^2 + 4}\right)}, \quad x \geqslant 0$$

The finite region S, shown shaded in Figure 1, is bounded by the curve, the x-axis and the line $x = 2$

The region S is rotated 360° about the x-axis.

Use integration to find the exact value of the volume of the solid generated, giving your answer in the form $k \ln a$, where k and a are constants.

(5)

$$\text{Volume} = \pi \int y^2 dx = \pi \int_0^2 \frac{2x}{3x^2+4} dx = \frac{1}{3}\pi \int_0^2 \frac{6x}{3x^2+4} dx$$

$$= \frac{1}{3}\pi \left[\ln(3x^2+4) \right]_0^2 = \frac{1}{3}\pi (\ln 16 - \ln 4)$$

$$= \frac{1}{3}\pi (\ln 4) = \frac{1}{3}\pi (2 \ln 2) = \frac{2}{3}\pi \ln 2$$

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5.

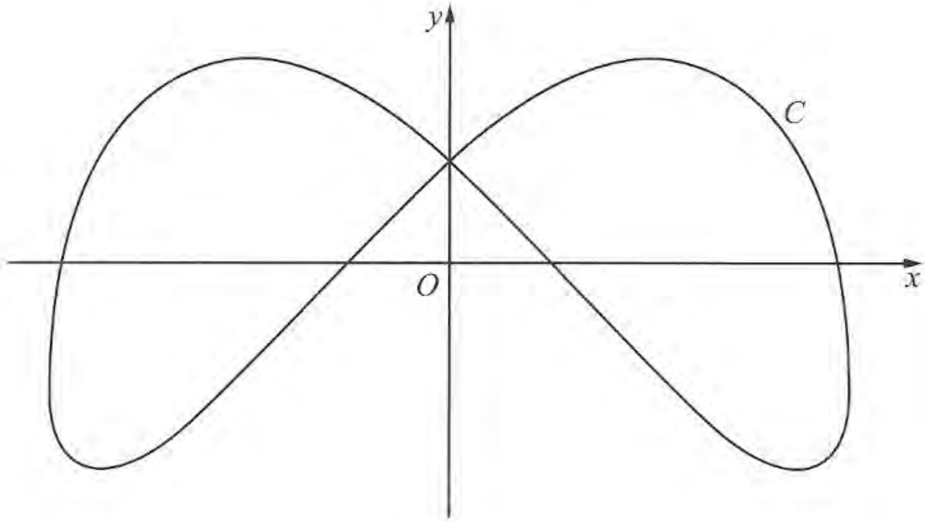


Figure 2

Figure 2 shows a sketch of the curve C with parametric equations

$$x = 4\sin\left(t + \frac{\pi}{6}\right), \quad y = 3\cos 2t, \quad 0 \leqslant t < 2\pi$$

$0 \leq 2t < 4\pi$

(a) Find an expression for $\frac{dy}{dx}$ in terms of t.

(3)

(b) Find the coordinates of all the points on C where $\frac{dy}{dx} = 0$

(5)

$\frac{dx}{dt} = 4\cos\left(t + \frac{\pi}{6}\right) \qquad \frac{dy}{dt} = -6\sin 2t$

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{-6\sin 2t}{4\cos\left(t + \frac{\pi}{6}\right)}$

b) $\frac{dy}{dx} = 0 \Rightarrow -6\sin 2t = 0 \Rightarrow \sin 2t = 0$

$2t = 0, \pi, 2\pi, 3\pi, \Rightarrow t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

$t = 0 \ (2, 3) \quad t = \frac{\pi}{2} \ (2\sqrt{3}, -3) \quad t = \pi \ (-2, 3) \quad t = \frac{3\pi}{2} \ (-2\sqrt{3}, -3)$

6.

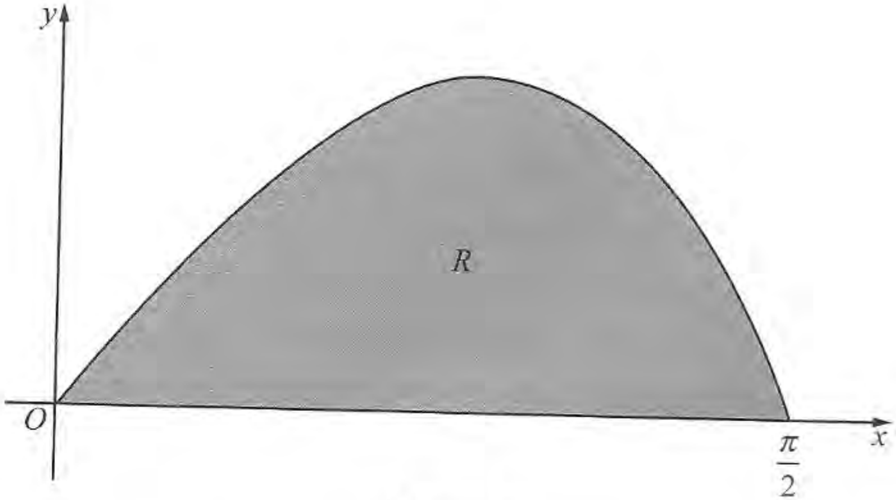


Figure 3

Figure 3 shows a sketch of the curve with equation $y = \frac{2\sin 2x}{(1+\cos x)}$, $0 \leqslant x \leqslant \frac{\pi}{2}$.

The finite region R, shown shaded in Figure 3, is bounded by the curve and the x-axis.

The table below shows corresponding values of x and y for $y = \frac{2\sin 2x}{(1+\cos x)}$.

x	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$
y	0	0.73508	1.17157	1.02280	0

(a) Complete the table above giving the missing value of y to 5 decimal places.

(1)

(b) Use the trapezium rule, with all the values of y in the completed table, to obtain an estimate for the area of R, giving your answer to 4 decimal places.

(3)

(c) Using the substitution $u = 1 + \cos x$, or otherwise, show that

$$\int \frac{2\sin 2x}{(1+\cos x)}\,dx = 4\ln(1+\cos x) - 4\cos x + k$$

where k is a constant.

(5)

(d) Hence calculate the error of the estimate in part (b), giving your answer to 2 significant figures.

(3)

$$\text{Area} = \cancel{\tfrac{1}{2}}\left(\frac{\pi}{8}\right)\left[0+0+\cancel{2}\left(0.73508+1.17157+1.02280\right)\right]$$

$$\simeq 1.1504 \text{ (4dp)}$$

c) $u = 1+\cos x$ $\qquad \frac{du}{dx} = -\sin x \Rightarrow dx = \frac{-du}{\sin x}$

$2\sin 2x = 4\sin x\cos x \qquad \cos x = u-1$

$$\therefore \int \frac{2\sin 2x}{1+\cos x}\,dx = 4\int \frac{\cancel{\sin x}(u-1)}{u} \cdot -\frac{du}{\cancel{\sin x}}$$

$$= -4\int \frac{u-1}{u}\,du = -4\int 1-\frac{1}{u}\,du$$

$$= -4\left[u-\ln u + c\right]$$

$$= 4\ln u - 4u + k \quad = 4\ln(1+\cos x) - 4(1+\cos x) + k$$

d) $4\left[\ln(1+\cos x) - (1+\cos x)\right]_0^{\frac{\pi}{2}}$

$$= 4\left[\left(\ln(\cancel{1+0}) - (1+0)\right) - \left(\ln(1+1) - (1+1)\right)\right]$$

$$= 4\left[1-\ln 2\right] \simeq 1.22741127 8\ldots$$

$\therefore$ error $= -0.077$ (2sf)

(Note - sign probably doesn't matter)

7. Relative to a fixed origin O, the point A has position vector $(2\mathbf{i} - \mathbf{j} + 5\mathbf{k})$,
the point B has position vector $(5\mathbf{i} + 2\mathbf{j} + 10\mathbf{k})$,
and the point D has position vector $(-\mathbf{i} + \mathbf{j} + 4\mathbf{k})$.

The line l passes through the points A and B.

(a) Find the vector $\overrightarrow{AB}$.

(2)

(b) Find a vector equation for the line l.

(2)

(c) Show that the size of the angle BAD is 109°, to the nearest degree.

(4)

The points A, B and D, together with a point C, are the vertices of the parallelogram $ABCD$, where $\overrightarrow{AB} = \overrightarrow{DC}$.

(d) Find the position vector of C.

(2)

(e) Find the area of the parallelogram $ABCD$, giving your answer to 3 significant figures.

(3)

(f) Find the shortest distance from the point D to the line l, giving your answer to 3 significant figures.

(2)

$a = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} \quad b = \begin{pmatrix} 5 \\ 2 \\ 10 \end{pmatrix} \quad d = \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}$

$\overrightarrow{AB} = b - a = \begin{pmatrix} 3 \\ 3 \\ 5 \end{pmatrix} \qquad l = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} + t\begin{pmatrix} 3 \\ 3 \\ 5 \end{pmatrix}$

c) $B\hat{A}D = \cos^{-1}\left(\dfrac{\overrightarrow{AB} \cdot \overrightarrow{AD}}{|AB||AD|}\right) \qquad \overrightarrow{AD} = \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix}$

$|AB| = \sqrt{3^2+3^2+5^2} = \sqrt{43}$

$B\hat{A}D = \cos^{-1}\left(\dfrac{-8}{\sqrt{43}\sqrt{14}}\right)$

$|AD| = \sqrt{3^2+2^2+1^2} = \sqrt{14}$

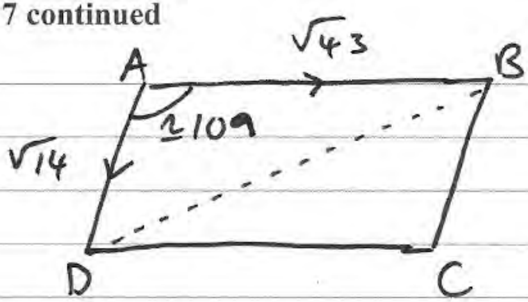
$AB \cdot AD = \begin{pmatrix} 3 \\ 3 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix}$

$= 109.0295\ldots$

$= -9 + 6 - 5 = \underline{-8}$

$\simeq 109°$ (nd)

Question 7 continued



$c = d + \vec{AB}$

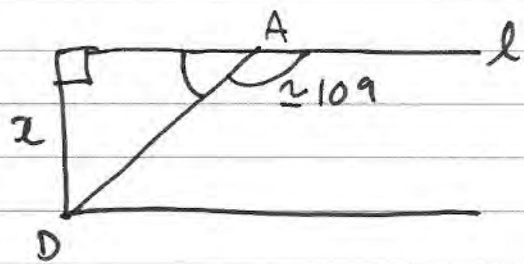
$c = \begin{pmatrix}-1\\1\\4\end{pmatrix} + \begin{pmatrix}3\\3\\5\end{pmatrix} = \begin{pmatrix}2\\4\\9\end{pmatrix}$

e) Area ABCD $= 2 \times$ Area $\triangle$BAD

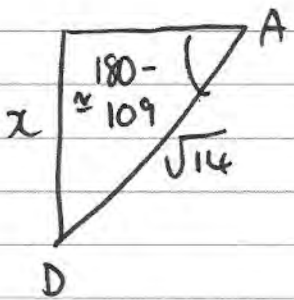
$= \cancel{2} \times \frac{1}{\cancel{2}}(\sqrt{43})(\sqrt{14})\sin(109.029\ldots)$

$\simeq 23.2$ (3sf)

f)



x = shortest distance l to D.



$x = \sqrt{14}\sin(180 - 109.029\ldots)$

$\simeq 3.54$ (3sf)

8. (a) Express $\dfrac{1}{P(5-P)}$ in partial fractions.

(3)

A team of conservationists is studying the population of meerkats on a nature reserve. The population is modelled by the differential equation

$$\frac{dP}{dt} = \frac{1}{15}P(5-P), \quad t \geqslant 0$$

where P, in thousands, is the population of meerkats and t is the time measured in years since the study began.

Given that when $t = 0$, $P = 1$,

(b) solve the differential equation, giving your answer in the form,

$$P = \frac{a}{b + c\mathrm{e}^{-\frac{1}{3}t}}$$

where a, b and c are integers.

(8)

(c) Hence show that the population cannot exceed 5000

(1)

a) $\dfrac{1}{P(5-P)} = \dfrac{A}{P} + \dfrac{B}{5-P} \Rightarrow 1 = A(5-P) + B(P)$

$P = 5 \Rightarrow 1 = 5B \quad B = \frac{1}{5}$

$P = 0 \Rightarrow 1 = 5A \quad A = \frac{1}{5}$

$\therefore \dfrac{1}{5P} + \dfrac{1}{5(5-P)}$

b) $\displaystyle\int \frac{1}{P(5-P)}\,dP = \int \frac{1}{15}\,dt \Rightarrow \frac{1}{5}\int \frac{1}{P} + \frac{1}{5-P}\,dP = \frac{1}{15}\int 1\,dt$

$\Rightarrow \displaystyle\int \frac{1}{P} + \frac{1}{5-P}\,dp = \frac{1}{3}\int 1\,dt \Rightarrow \ln P - \ln(5-P) = \frac{1}{3}t + c$

$\Rightarrow \ln\left(\dfrac{P}{5-P}\right) = \frac{1}{3}t + c$

$t=0, P=1 \Rightarrow \ln\left(\frac{1}{4}\right) = c \Rightarrow c = -\ln 4$

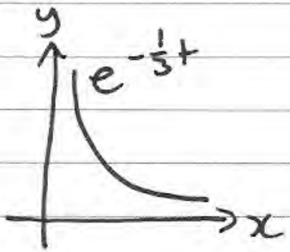
$\Rightarrow \ln\left(\dfrac{P}{5-P}\right) = \frac{1}{3}t - \ln 4$

$$\frac{P}{5-P} = e^{\frac{1}{3}t - \ln 4} = e^{\frac{1}{3}t} \div e^{\ln 4} = \frac{e^{\frac{1}{3}t}}{4}$$

$$\Rightarrow 4P = 5e^{\frac{1}{3}t} - Pe^{\frac{1}{3}t} \Rightarrow P\left(4+e^{\frac{1}{3}t}\right) = 5e^{\frac{1}{3}t}$$

$$\Rightarrow P = \frac{5e^{\frac{1}{3}t}}{4+e^{\frac{1}{3}t}} \quad \begin{matrix} \div e^{\frac{1}{3}t} \\ \div e^{\frac{1}{3}t} \end{matrix} \quad \Rightarrow P = \frac{5}{4e^{-\frac{1}{3}t}+1}$$

$\underline{a = 5}$ $\underline{b = 1}$ $\underline{c = 4}$



as $x \to \infty$ $e^{-\frac{1}{3}t} \to 0$

$$\Rightarrow 4e^{-\frac{1}{3}t} + 1 \to 1$$

$$4e^{-\frac{1}{3}t} + 1 > 1$$

$$\therefore P \to \frac{5}{1} \to 5$$

$$P < 5$$

P is in thousands $\therefore$ Population can not exceed 5000