C3 DIFFERENTIATION — Answers - Worksheet K

1 **a** $\frac{dy}{dx} = -\frac{1}{4}x^{-2} - \frac{1}{x}$

$x = 1 \;\therefore\;$ grad $= -\frac{5}{4}$

b grad of normal $= \frac{4}{5}$

$\therefore\; y - \frac{1}{4} = \frac{4}{5}(x - 1)$

$16x - 20y - 11 = 0$

2 **a** $\frac{dy}{dx} = 1 \times e^{-2x} + x \times (-2e^{-2x})$

$= e^{-2x}(1 - 2x)$

$\frac{d^2y}{dx^2} = -2e^{-2x} \times (1 - 2x) + e^{-2x} \times (-2)$

$= 4e^{-2x}(x - 1)$

b SP: $e^{-2x}(1 - 2x) = 0$

$x = \frac{1}{2}$

$\therefore\; (\frac{1}{2}, \frac{1}{2}e^{-1})$

when $x = \frac{1}{2}$, $\frac{d^2y}{dx^2} = -2e^{-1}$

$\frac{d^2y}{dx^2} < 0 \;\therefore\;$ maximum

3 **a** $y = 0 \Rightarrow x = \sqrt{3}$

$\therefore\; (\sqrt{3}, 0)$

b $= \frac{1}{2}(e^y + 2)^{-\frac{1}{2}} \times e^y$

$= \frac{e^y}{2\sqrt{e^y + 2}}$

c $\frac{dy}{dx} = 1 \div \frac{dx}{dy} = \frac{2\sqrt{e^y + 2}}{e^y}$

grad $= 2\sqrt{3}$

$\therefore\; y - 0 = 2\sqrt{3}(x - \sqrt{3})$

at Q, $x = 0 \;\therefore\; y = -6$

area $= \frac{1}{2} \times \sqrt{3} \times 6 = 3\sqrt{3}$

4 **a** $t = 0$, $m = 680$

$t = 100$, $m = 653.63$

% red'n $= \frac{680 - 653.63}{680} \times 100\% = 3.88\%$ (3sf)

b $640 = 600 + 80e^{-0.004t}$

$t = \frac{-1}{0.004}\ln\frac{1}{2} = 173$ (3sf)

c $\frac{dm}{dt} = 80 \times (-0.004)e^{-0.004t} = -0.32e^{-0.004t}$

$t = 150$, $\frac{dm}{dt} = -0.176$

$\therefore$ mass decreasing at 0.176 g yr^{-1} (3sf)

5 **a** $= \frac{1}{2}(\sin x + \cos x)^{-\frac{1}{2}} \times (\cos x - \sin x)$

$= \frac{\cos x - \sin x}{2\sqrt{\sin x + \cos x}}$

b $= \frac{d}{dx}[\ln(x - 1) - \ln(2x + 1)]$

$= \frac{1}{x - 1} - \frac{1}{2x + 1} \times 2$

$= \frac{1}{x - 1} - \frac{2}{2x + 1}$

6 **a** $\frac{dy}{dx} = 5(2x - 3)^4 \times 2 = 10(2x - 3)^4$

$x = 1 \;\therefore\;$ grad $= 10$

$\therefore\; y + 1 = 10(x - 1)$

$[\, y = 10x - 11 \,]$

b at Q $\quad 10(2x - 3)^4 = 10$

$2x - 3 = \pm 1$

$x = 1$ (at P) or 2

$\therefore\; Q\,(2, 1)$

7 **a** $\frac{dy}{dx} = -2(x^2 - 5)^{-2} \times 2x = \frac{-4x}{(x^2-5)^2}$

SP: $\frac{-4x}{(x^2-5)^2} = 0$

$x = 0$

$\therefore\ (0, -\frac{2}{5})$

b $x = 3, y = \frac{1}{2}$

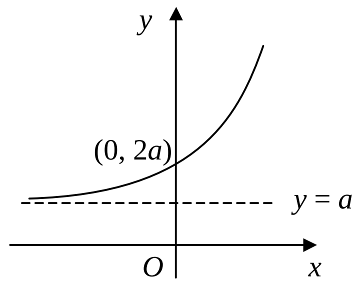
grad $= -\frac{3}{4}$

$\therefore\ y - \frac{1}{2} = -\frac{3}{4}(x - 3)$

$4y - 2 = -3x + 9$

$3x + 4y - 11 = 0$

8 **a**



b $y = ae^x + a$

swap $x = ae^y + a$

$y = \ln \frac{x-a}{a}$

$f^{-1} : x \to \ln \frac{x-a}{a},\ x \in \mathbb{R},\ x > a$

c $x = 1\ \therefore\ y = ae + a$

$f'(x) = ae^x$, grad $= ae$

$\therefore\ y - (ae + a) = ae(x - 1)$

$[\ y = aex + a\ ]$

9 **a** $\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)$

$= \frac{-\sin x \times \sin x - \cos x \times \cos x}{\sin^2 x}$

$= -\frac{\sin^2 x + \cos^2 x}{\sin^2 x}$

$= -\frac{1}{\sin^2 x} = -\text{cosec}^2\, x$

b $\frac{dy}{dx} = e^x \times \cot x + e^x \times (-\text{cosec}^2\, x)$

$= e^x(\cot x - \text{cosec}^2\, x)$

SP: $e^x(\cot x - \text{cosec}^2\, x) = 0$

$e^x \neq 0\ \therefore\ \cot x = \text{cosec}^2\, x$

$\frac{\cos x}{\sin x} = \frac{1}{\sin^2 x}$

$\sin x \cos x = 1$

$\sin 2x = 2$

$|\sin 2x| \leq 1\ \therefore$ no solutions

$\therefore$ no turning points

10 **a** $\frac{dy}{dx} = 3(2 + \ln x)^2 \times \frac{1}{x}$

$= \frac{3}{x}(2 + \ln x)^2$

b SP: $\frac{3}{x}(2 + \ln x)^2 = 0$

$\ln x = -2$

$x = e^{-2}$

$\therefore\ (e^{-2}, 0)$

c $x = e,\ y = 27$

grad $= \frac{27}{e}$

$\therefore\ y - 27 = \frac{27}{e}(x - e)$

$y = \frac{27}{e}x$

when $x = 0,\ y = 0$

$\therefore$ passes through origin

11 **a** $= \dfrac{1}{9 - x^2} \times (-2x)$

$= \dfrac{-2x}{9 - x^2}$

b SP: $\dfrac{-2x}{9 - x^2} = 0$

$x = 0$

$\therefore$ (0, ln 9)

c $x = 1$, $y = \ln 8 = \ln 2^3 = 3 \ln 2$

grad $= -\frac{1}{4}$

$\therefore$ grad of normal = 4

$\therefore$ $y - 3 \ln 2 = 4(x - 1)$

$y = 4x - 4 + 3 \ln 2$

12 **a** model A: $t = 3$, $M = 764$ (3sf)

model B: $t = 3$, $M = 732$ (3sf)

b model A:

$\dfrac{\mathrm{d}M}{\mathrm{d}t} = 1500(3t + 2)^{-2} \times 3 = \dfrac{4500}{(3t+2)^2}$

$t = 3$, $\dfrac{\mathrm{d}M}{\mathrm{d}t} = 37.2$

$\therefore$ increasing at 37.2 tonnes yr^{-1} (3sf)

model B:

$\dfrac{\mathrm{d}M}{\mathrm{d}t} = 1500[2 + 5 \ln (t + 1)]^{-2} \times \dfrac{5}{t+1}$

$= \dfrac{7500}{(t+1)[2+5\ln(t+1)]^2}$

$t = 3$, $\dfrac{\mathrm{d}M}{\mathrm{d}t} = 23.5$

$\therefore$ increasing at 23.5 tonnes yr^{-1} (3sf)