Algebraic Fractions

An algebraic fraction can always be expressed in different, yet equivalent forms. A fraction is expressed in its simplest form by cancelling any factors which are common to both the numerator and the denominator.

Algebraic Fractions can be simplified by cancelling down. To do this, numerators and denominators must be fully factorised first. If there are fractions within the numerator/denominator, multiply by a common factor to get rid of these and create an equivalent fraction:

$$\frac{\frac{1}{2}x+1}{\frac{1}{3}x+\frac{2}{3}} = \frac{(\frac{1}{2}x+1)\times 6}{(\frac{1}{3}x+\frac{2}{3})\times 6} = \frac{3x+6}{2x+4} = \frac{3(x+2)}{2(x+2)} = \frac{3}{2}$$

To multiply fractions, simply multiply the numerators and multiply the denominators. If possible, cancel down first. To divide by a fraction, multiply by the reciprocal of the fraction:

$$\frac{x+1}{2}\times\frac{3}{x^2-1} = \frac{x+1}{2}\times\frac{3}{(x+1)(x-1)} = \frac{3(x+1)}{2(x+1)(x-1)} = \frac{3}{2(x+1)}$$

To add or subtract fractions, they must have the same denominator. This is done by finding the lowest common multiple of the denominators:

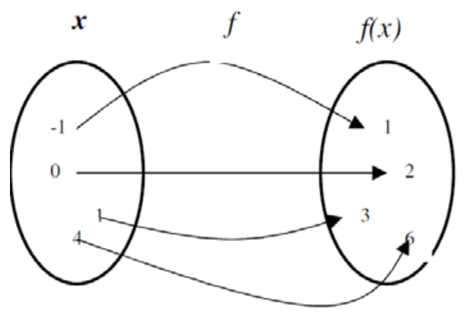
$$\frac{1}{(x+1)}+\frac{4}{(x+6)} = \frac{1(x+6)}{(x+1)(x+6)}+\frac{4(x+1)}{(x+6)(x+1)} = \frac{(x+6)+4(x+1)}{(x+1)(x+6)} = \frac{5x+10}{(x+1)(x+6)}$$

When the numerator has the same or a higher degree than the denominator (it is an improper fraction), you can divide the terms to produce a mixed fraction:

$$\begin{array}{r|rrrr}
 & & 8x^2 & -2x & -3 \\ \hline
x-1 & 8x^3 & -10x^2 & -x & +3 \\
 & -8x^3 & +8x^2 & & \\ \hline
 & & -2x^2 & -x & \\
 & & 2x^2 & -2x & \\ \hline
 & & & -3x & +3 \\
 & & & 3x & -3 \\ \hline
 & & & & 0
\end{array}$$

Functions

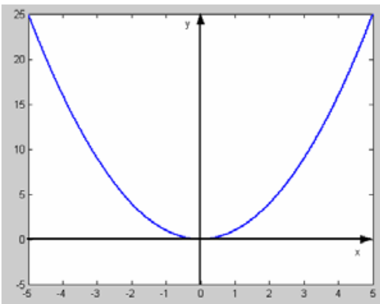
Functions are special types of mappings such that every element of the domain is mapped to exactly one element in the range. This is illustrated below for the function f (x) = x + 2



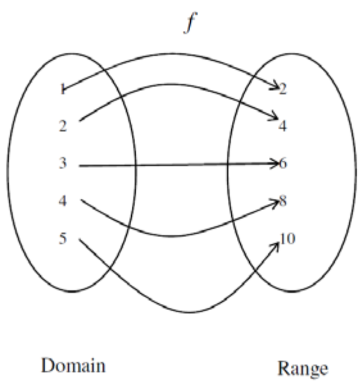
For example, f 'sends' -1 to 1 (or -1 is mapped to 1), f 'sends' 4 to 6 (or 4 is mapped to 6) etc.
We say that '1 is the image of -1 under f', '6 is the image of 4 under f' etc.

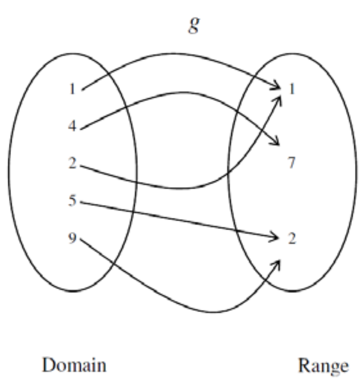
The set of all numbers that we can feed into a function is called the domain of the function. The set of all numbers that the function produces is called the range of a function. Often when dealing with simple algebraic function, such as f (x) = x + 2, we take the domain of the function to be the set of real numbers, $\mathbb{R}$. In other words, we can feed in any real number x into the function and it will give us a (real) number out. Sometimes we restrict the domain, for example we may wish to consider the function f(x) = x + 2 in the interval -2 < x < 2.

Consider the function f (x) = x^2. What is the range of f (x)? Are there any restrictions on the values that this function can produce? When trying to work out the range of a function it is often useful to consider the graph of the function, this is shown left. We can see that the function only gives out positive numbers (x^2 is always positive for any real number x). There are no further restrictions. We can see that f can take any positive value, therefore the range of f is the set of all positive numbers, we may write f(x) $\geq$ 0.

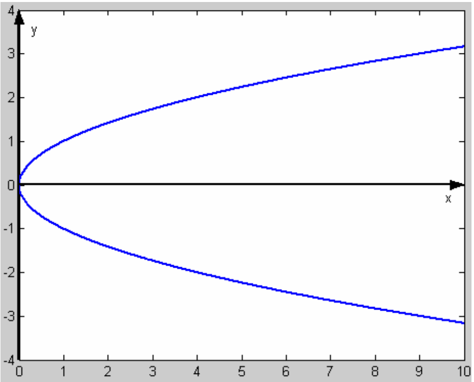


When each of the elements of the domain is mapped to a unique element of the range, under a mapping, the mapping is said to be **one-to-one**. When two or more elements of the domain are mapped to the same element of the range under a mapping, the mapping is said to be **many-to-one**. Below are two examples.
The mapping f is one-to-one, the mapping g is many-to-one.
We need to define more precisely what we mean by a 'function'.



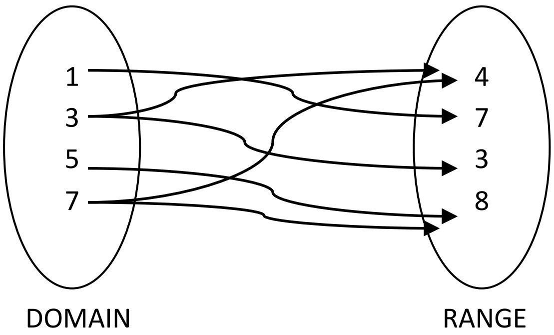


We can define a function as a rule that uniquely associates each and every member of one set with a member or members of another set. This means that every element of the domain is mapped to an element of the range such that the image of any element in the domain is unique. In other words, each and every element of the domain must be mapped to one and only one element of the range. For example, consider the expression $y = \pm\sqrt{x}$.

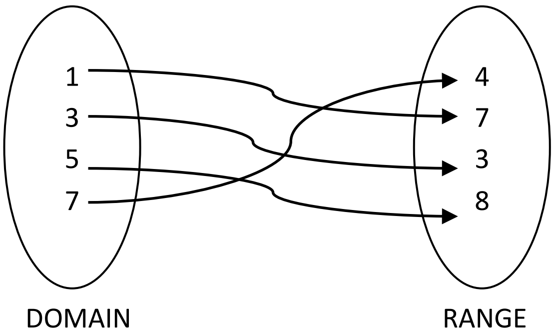


Notice that any value of x in the domain, except x = 0 , (i.e. any positive real number) is mapped to two different values in the range. Therefore $y = \pm\sqrt{x}$ is **not** a function.

When looking for the domain of a function, look out for values that would leave a negative root or 0 on the bottom of a fraction. At these values of x, y is undefined.



not a function



a function

Many mappings can be made into functions by changing the domain. For example, the 'root of x' mapping can be made into a function by changing the domain from all real numbers, to all positive numbers. This will cut off the bottom half of the graph, meaning every element in the domain is matched uniquely to an element in the range.

Composite Functions

Consider the function, g (x) = $(x - 2)^2$. If we were given a set of numbers and asked to perform the function g on each of them, we would have to carry out two separate calculations on any one of the given numbers; first we would have to subtract two from the number, then we would square the result. Thus, we may think of the function g as two functions in one. g is composed of the functions p(x) = x - 2 and
q(x) = $(p(x))^2$. We say that g is a composite function, and we write g (x) = q(x) × p(x).

fg(x) means 'apply g first, then f '

Inverse Functions

Consider the simple, linear function $f(x) = 3x - 27$. If we feed $x = 2$ into this function, we get out $f(2) = -21$. Suppose that we are told that the function has produced the number 9, but we do not know what input produced this number. We can easily work out the input number:

$$f(n) = 3n - 27 = 9 \quad \rightarrow \quad n = \frac{9 + 27}{3} = 12$$

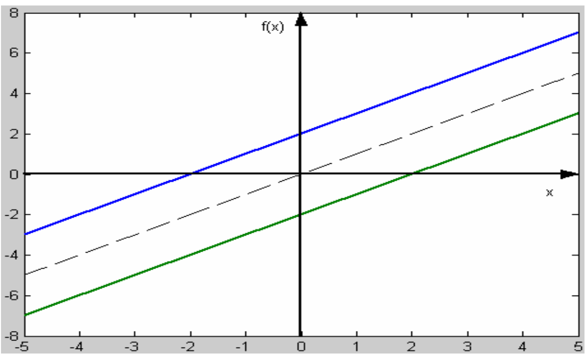
If we know the output of a given function and we require the input of the function, we can find it by using the **inverse function**.

The inverse of a function f(x) is written $f^{-1}(x)$, and performs the opposite operation(s) to the function. There are two methods to find the inverse of a function:

- Work Backwards
  - We can think of a 'function machine' which takes an input, performs the function on it and produces an output. The inverse function machine takes the output from the original function and gives us the original input number.

- Change the Subject
  - Let y = f(x), and then rearrange the formulae to find x. You then swap x and y (y being f(x)). For example, g(x) = 4x – 3, so let y = 4x – 3.

    Rearranged to find x gives $f(x) = \frac{x+3}{4}$



We can see that the graph of $f^{-1}(x)$ is a reflection of the graph of $f(x)$ in the line $y = x$. In fact, this is a general result for any invertible function (a function that has an inverse).
Note that not all functions are invertible. **Only one-to-one functions are invertible.**

The domain is the set of all numbers where the function is defined. Eg, $f(x) = {}^{1}/_{x}$ is defined everywhere except at x=0. The range is the set of all possible values the function can take (it usually helps to sketch a graph. So for example, the range of $f(x) = x^2$ is x>0.

The range of $f(x)$ is the domain of $f^{-1}(x)$, and the domain of $f(x)$ is the range of $f^{-1}(x)$.
The x and y coordinates where a graph meets the axis swap for the inverse graph.

If f^{-1} exists, then $f^{-1}f(x) = ff^{-1}(x) = x$.

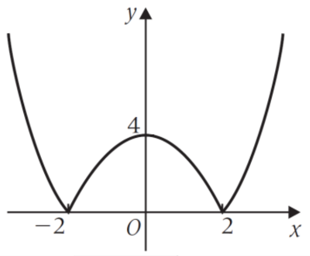
The Modulus Function

The modulus sign indicates that we take the absolute value of the expression inside the modulus sign, i.e. all values are positive.
e.g. $|2 - 3| = 1$, $|0 - 5| = 5$, $|-2| = 2$, $|1 + 7| = 8$.
We can define:

$$|x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

$\underline{y = |f(x)|}$

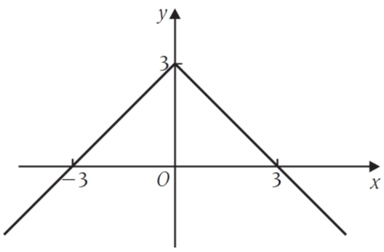
Let us consider the graph of $y = |x^2 - 4|$. As $|x^2 - 4|$ is always positive, the graph of $y = |x^2 - 4|$ cannot exist below the x-axis. For positive values of x, the graph of $|x^2 - 4|$ is the same as the graph of $y = |x^2 - 4|$; but for negative values of x, the graph $y = |x^2 - 4|$ is the line $y = -(x^2 - 4)$



Note that the graph of $y = |x^2 - 4|$is similar to the graph of $y = x^2 - 4$ except that the negative region of the graph is reflected in the x-axis. In general, the graph of $y = |f(x)|$ is similar to the graph of $y = f(x)$ except that the negative region of the graph is reflected in the x-axis.

$\underline{y = f(|x|)}$

For the function $y = f(|x|)$, the value of y at, for example x= -3, is the same as the value at y for x=3. This is because $y = f(|-3|) = f(|3|)$. The result of this is that the graph is reflected in the y-axis. The example below is of $y = 3 - |x|$



There are two ways to sketch graphs of the form $y =$ $f(|x|)$:

1. Draw the graph for positive value of x, then reflect it in the y-axis to give a line of symmetry at x=0
2. Draw the graph of $|x|$ and then transform it by the vector $\binom{0}{3}$

To solve an equation of the type $|f(x)| = g(x)$:

- *Use a sketch to locate where the roots should roughly lie*
- *Solve algebraically, using $-f(x)$ for reflected parts of $y = f(x)$ and $-g(x)$ for reflected parts of $y = g(x)$*

To solve the equation $|ax + b| = |cx + d|$ it is easiest to square both sides to remove the modulus sign.
Equations of the form $|ax + b| = cx + d$ need to be solved graphically.

Transformations

When given a sketch of $y = f(x)$, you need to be able to sketch transformations of the graph, showing coordinates of the points to which the given points are mapped.

Transformation	Description
$y = \mathrm{f}(x) + a \qquad a > 0$	Translation of $y = \mathrm{f}(x)$ through $\begin{pmatrix} 0 \\ a \end{pmatrix}$
$y = \mathrm{f}(x + a) \qquad a > 0$	Translation of $y = \mathrm{f}(x)$ through $\begin{pmatrix} -a \\ 0 \end{pmatrix}$
$y = a\mathrm{f}(x) \qquad a > 0$	Stretch of $y = \mathrm{f}(x)$ parallel to y-axis with scale factor a
$y = \mathrm{f}(ax) \qquad a > 0$	Stretch of $y = \mathrm{f}(x)$ parallel to x-axis with scale factor $\frac{1}{a}$
$y = \lvert\mathrm{f}(x)\rvert$	For $y \geq 0$, sketch $y = \mathrm{f}(x)$ For $y < 0$, reflect $y = \mathrm{f}(x)$ in the x-axis
$y = \mathrm{f}(\lvert x\rvert)$	For $x \geq 0$, sketch $y = \mathrm{f}(x)$ For $x < 0$, reflect [$y = \mathrm{f}(x)$ for $x > 0$] in the y-axis
Also useful	
$y = -\mathrm{f}(x)$	Reflection of $y = \mathrm{f}(x)$ in the x-axis (line $y = 0$)
$y = \mathrm{f}(-x)$	Reflection of $y = \mathrm{f}(x)$ in the y-axis (line $x = 0$)

These may be combined to give, for example $bf(x + a)$, which is a horizontal translation of –a followed by a vertical stretch of scale factor b.

For combinations of transformations, the graph can be built up one step at a time starting from a given curve.

Numerical Methods

In real life situations, we are often faced with equations which have no analytic solution. That is to say we cannot find an *exact* solution to the equation. For example, we can solve the equation $x^2 + x - 2 = 0$ by factorising $(x + 2)(x - 1) = 0 \rightarrow x = -2$ or $x = 1$.

The above equation can be solved analytically to find the exact solutions. What about the equation $cos\ (x) - x = 0$. This equation cannot be solved analytically unlike the previous example. We cannot find the exact solution of this equation using algebraic, or any other techniques. We cannot solve this equation exactly, however we can find the approximate solution or solutions to the equation $cos\ (x) - x = 0$. In fact, we can find the solution or solutions to an arbitrary degree of accuracy, however the more accurate we require our solution(s), the longer the process.

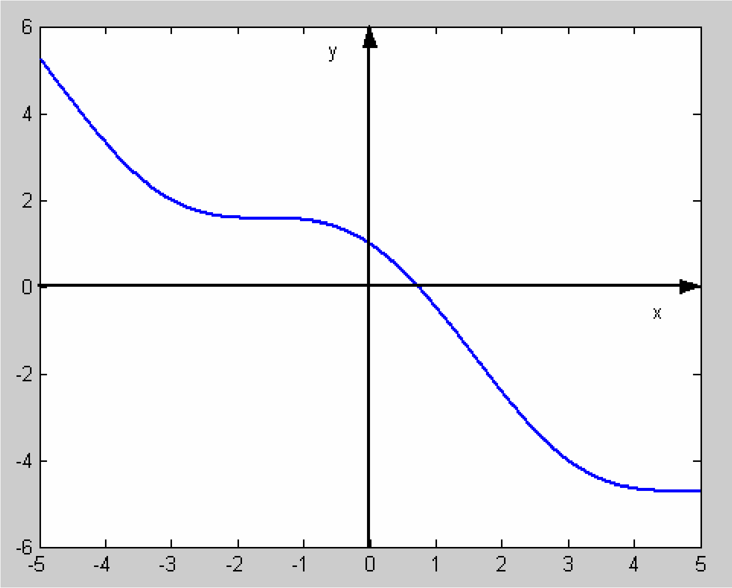
There are 3 numerical methods that can be used to find approximations to the root(s) of a function which may not be possible to find analytically:

1. Graphically
2. Looking for a change of sign
3. Iteration

Graphically

You can find approximations for the roots of the equation $f(x) = 0$ graphically. This is the simplest numerical method, and is done by drawing the graph of the given function. The root(s) will lie where the graph crosses the x-axis. From looking at a graph, you can see roughly where this point is, therefore can say a root lies between $x = a$ and $x = b$

Suppose we want to solve the equation $cos\, x - x = 0$. The graph of $y = f\,(x) = cos\, x - x$ is shown. We know that the solution of $f\,(x) = 0$ corresponds to the point where the graph of $f\,(x)$ cuts the *x*-axis. So we can tell, just from plotting the graph, that the solution is somewhere around $x = 0.7$.
We notice that to the left of the root, the function is positive and to the right of the root the function is negative.



Change of Sign

In general, the sign of a function, (x), to the left of a root is opposite to the sign of the function to the right of the root. We can use this simple fact to help us find the roots of equations.

For example to solve $cos\,x - x = 0$, we can calculate the value of the function at a few points and see if we get a change of sign: $f\,(0) = cos(0) - 0 = 1 - 0 = 1$ (positive), $f\,(1) = cos(1) - 1$ » $- 0.4597$ (negative), so we can say that there is a zero somewhere between x = 0 and x =1.

To get a more accurate approximation to the root, we could look at the value of the function $f\,(x)$ at the point mid-way between $x = 0$ and $x = 1$, i.e. at the point $x = 0.5$. We see that $f\,(0.5) = cos\,(0.5) - 0.5 = 0.3776$ (positive). So now we can say that the root lies somewhere between $x = 0.5$ and $x = 1$. To get a better approximation, you can continue to values closer and closer together.

In general, if you find an interval in which $f(x)$ changes sign, then the interval must contain a root of the equation $f(x) = 0$

The only exception to this is when $f(x)$ has a discontinuity in the interval e.g. $f(x) = {}^1\!/_x$ has a discontinuity at $x = 0$.

Iteration

The iteration method uses a formula that by inputting an approximate value of x, a more accurate value is outputted.

The first step in the process is to rearrange the equation into the form x = some function of x (if not already given). Then an approximate value of x is input into the equation, and the process repeats to give more and more accurate values of the roots of the equation.

For example, let us consider the equation $\cos(x) - x = 0$. The most obvious way of rearranging this would give $x = \cos(x)$. Now the iteration formula is $x_{n+1} = \cos(x_n)$. We start with an initial guess to the root x_0. Let us make our initial guess x_0 = 0.7 . We then feed the initial guess into the iteration formula, to produce a better approximation of the solution, x_1 . We then feed x_1 into the iteration formula to produce a better approximation, x_2 and so forth.

So, $x_1 = \cos x_0$ with initial guess $x_0 = 0.7$, we get $x_1 = \cos(0.7) = 0.7648$.

$x_2 = \cos x_1 = \cos(0.7648) = 0.7215$

$x_3 = \cos x_2 = \cos(0.7215) = 0.7508$

$x_4 = \cos x_3 = \cos(0.7508) = 0.7311$

$x_4 = \cos x_3 = \cos(0.7508) = 0.7311$

$x_5 = \cos x_4 = \cos(0.7311) = 0.7444$

$x_6 = \cos x_5 = \cos(0.7344) = 0.7422$

$x_8 = \cos x_7 = \cos(0.7370) = 0.7405$

$x_9 = \cos x_8 = \cos(0.7405) = 0.7381$

$x_{10} = \cos x_9 = \cos(0.7381) = 0.7397$

$x_{11} = \cos x_{10} = \cos(0.7397) = 0.7387$

$x_{12} = \cos x_{11} = \cos(0.7387) = 0.7394$

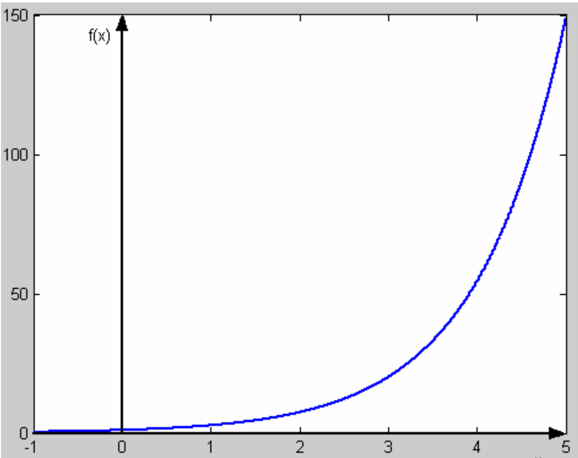
$$\boxed{x \approx 0.7394}$$

Exponential and Log Functions

The exponential function (e) and the natural logarithm function (ln) are both the inverse operations of one another. $e^{\ln(x)} = \ln(e^x) = x$

e is a special number similar to π. It has a value of 2.718 to 3dp (although it is an irrational number). This value is the only one at which the value of the gradient of an exponential graph at a given point is equal to the gradient. As with all exponential functions, it passes through the point (0, 1), providing it has not been transformed.

The domain is all real numbers and the range is f(x) > 0.

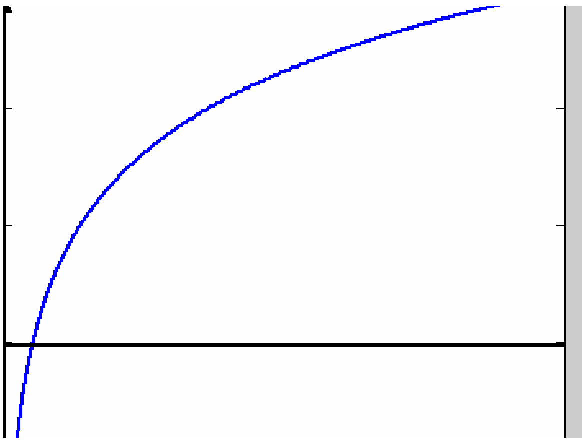


The inverse of the exponential function e^x is the logarithmic function base *e*, ln(x).

$$If\ f(x) = e^x, then\ f(x)^{-1} = ln\ x$$

The natural log function is a reflection of the line y = e^x in the line y = x. It passes through the point (1, 0) providing it has not been transformed. The main features of the graph are:

- As $x \to 0, y \to -\infty$
- ln(x) doesn't exist for negative numbers
- when x = 1, y = 0
- As $x \to \infty, y \to \infty$ (slowly)



The domain is all positive numbers. The range is all real numbers.

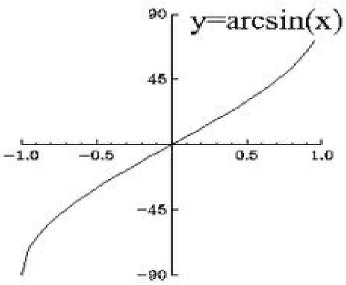
To solve an equation using ln(x) or e^x you must change the subject of the formula and use the fact that they are inverses of each other in order to find x.

Trigonometry

For one-to-one functions, you can draw its inverse graph by reflecting it in the line y = x. The three trig functions sin(x), cos(x) and tan(x) only have an inverse function if their domains are restricted so that they are one-to-one functions. The notations used for these inverse functions are arcsin(x), arcos(x) and arctan(x).

arcsin(x)

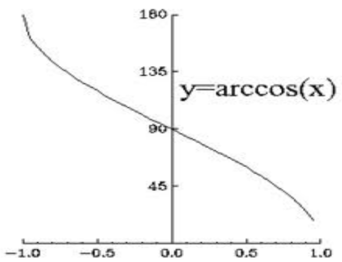
When drawing the graph of arcsin(x), you start with a normal sin(x) graph. The domain of this has to be restricted to $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ or $-90 \leq x \leq 90$. This makes it a one-to-one function, taking the range $-1 \leq \sin(x) \leq 1$.



The graph can then be reflected in the line y = x to find its inverse graph, ie. the graph of arcsin(x).
As with all inverse graphs, the range and the domain swap, so the domain is $-\frac{\pi}{2} \leq \arcsin(x) \leq \frac{\pi}{2}$ and the range becomes $-1 \leq x \leq 1$

arccos(x)

For arcos(x), the graph of cos(x) is restricted to a domain of $0 \leq x \leq \pi$ or $0 \leq x \leq 90$, and a range of $-1 \leq \cos(x) \leq 1$.



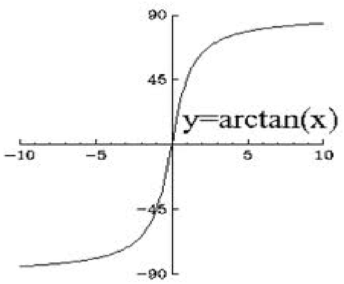
When reflected in the line y = x it gives the graph of arcos(x).

The domain becomes $-1 \leq x \leq 1$

The range becomes $0 \leq arcos(x) \leq \pi$

arctan(x)

By restricting the domain of tan(x) to $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, it becomes a one to one function. The range stays $\tan(x) \in \mathbb{R}$.



When reflected in the line y = x it gives the graph of arctan(x).

The domain becomes $x \in \mathbb{R}$

The range becomes $-\frac{\pi}{2} \leq \arctan(x) \leq \frac{\pi}{2}$

<u>nb:</u> *$sin^{-1}\, x$ does **not** mean the same thing as $\frac{1}{sin\, x}$*

Secant, Cosecant and Cotangent

In addition to the normal three trig functions, there are secant, cosecant and cotangent:

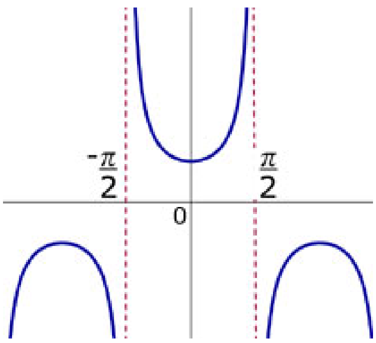
$$\sec(x) = \frac{1}{\cos(x)}$$

$$\operatorname{cosec}(x) = \frac{1}{\sin(x)}$$

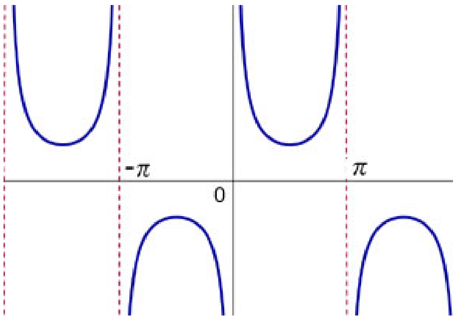
$$\cot(x) = \frac{1}{\tan(x)} = \frac{\cos(x)}{\sin(x)}$$

These are undefined when sin(x)/cos(x)/tan(x) is equal to zero, as it would leave 0 as the numerator, which is not a real number.

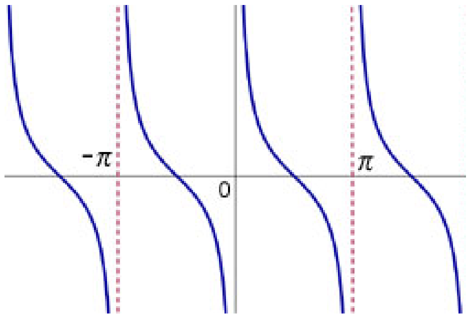
The graphs of these functions are found by plotting points using these formulae.



Secant



Cosecant



Cotangent

Pythagorean Identities

From $\sin^2\theta + \cos^2\theta \equiv 1$, we can work out two more identities using the above functions:

$$\sin^2\theta + \cos^2\theta \equiv 1$$

$$\Downarrow$$

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} \equiv \frac{1}{\cos^2\theta}$$

$$\Downarrow$$

$$tan^2\,\theta + 1 = sec^2\,\theta$$

$$\sin^2\theta + \cos^2\theta \equiv 1$$

$$\Downarrow$$

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} \equiv \frac{1}{\sin^2\theta}$$

$$\Downarrow$$

$$1 + cot^2\,\theta = cosec^2\theta$$

Addition Formulae

The addition formulae can help solving equations by removing an unknown from a bracket, or at least separate it from know values, eg. $\sin(\theta + 30)$
The addition formulae are:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

Double-Angle Formula

From the above addition formulae, the following can be worked out. They allow you to solve more equations and prove more identities.
The double-angle formulae are as follows:

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A \quad = \quad 2\cos^2 A - 1 \quad = \quad 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

The $cos(2A)$ formula has 3 different versions that can be worked out by substituting in the identity $\sin^2 A + \cos^2 A = 1$.

Factor Formulae

From the previously stated identities, these factor formulae can be worked out, however they are given in the formulae sheet.

$$\sin A \pm \sin B = 2 \sin \frac{1}{2}(A \pm B) \cos \frac{1}{2}(A \mp B)$$

$$\cos A + \cos B = 2 \cos \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B)$$

The R-Alpha Method

You can write expressions of the form $a\cos x + b\sin x$, where a and b are constants, as a sine function or a cosine function only. Having the ability to do this enables you to solve certain sorts of trigonometric equations and find maximum and minimum values of some trigonometric functions. This will only find values for acute angles of α.

We study the expression $Rcos(x-\alpha)$ and note that $cos(x-\alpha)$ can be expanded using an addition formula.

$$Rcos(x - \alpha) = R(cos\, x\cos\alpha + sin\, x\sin\alpha) = Rcos\, x\cos\alpha + Rsin\, x\, sin\,\alpha$$

We can re-order this expression as follows:

$$Rcos(x - \alpha) = (Rcos\,\alpha)\, cos\, x + (Rsin\,\alpha)\, sin\, x$$

If we want to write an expression of the form $a\, cos\, x + b\, sin\, x$ in the form $Rcos(x - \alpha)$, we can do this by comparing $a\, cos\, x + b\, sin\, x$ with $(Rcos\,\alpha)\, cos\, x + (Rsin\,\alpha)\, sin\, x$

Doing this we see that

$$a = Rcos\,\alpha \qquad (1)$$

and $$b = Rsin\,\alpha \qquad (2)$$

By squaring each of Equations (1) and (2) and adding we find:

$$\begin{aligned} a^2 + b^2 &= R^2 cos^2\alpha + R^2 sin^2\alpha \\ &= R^2(cos^2\alpha + sin^2\alpha) \\ &= R^2 \end{aligned}$$

Therefore $R = \sqrt{a^2 + b^2}$

We can find α by dividing Equation (2) by Equation (1) to give

$$\frac{Rsin\,\alpha}{Rcos\,\alpha} = \frac{b}{a} = \tan\alpha$$

Knowing $\tan\alpha$ we can find α by finding arctan. So, now we can write any expression of the form $a\cos x + b\sin x$ in the form $Rcos(x - \alpha)$.

This same method works to find solutions in the form $Rsin(x-\alpha)$. This is done by using the addition rule with $sin(x-\alpha)$ and then following the normal steps.

<u>Maximum/Minimum Points:</u>

The maximum value of the cosine function is 1 and this occurs when the angle x − α = 0, i.e. when x = α. The maximum value of a sine function is also 1, however it occurs when the angle x – α = 90, i.e. when x = 90 + α.

Conversely, the minimum value of the cosine and sine functions is -1. For cos, this occurs at 180. For sin it occurs at 270.

Using these facts, the maximum and minimum points, and the values of x at which they occur, can be worked out. However, the value of R before the sin/cos function will stretch the graphs, meaning the max/min point is generally $\pm R$. This needs to be taken into consideration when finding the values for x.

Differentiation

The Chain Rule

The chain rule exists for differentiating a function of another function, for example $\cos(x^2)$. This is an example of a composite function – it comprises of two functions f(x) and g(x), where f(x) = cos x and g(x) = x^2. Composite functions come in the form f(g(x)).

To differentiate y = f(g(x)), let u = g(x). Then y = f(u). The chain rule in order to differentiate is given by

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Also, a particular case of the chain rule is that

$$\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$$

The Product Rule

By using the product rule, you can differentiate functions that have been multiplied together., for example $x^2\sqrt{3x-1}$.
To differentiate the product of two functions, differentiate the first function, then multiply it by the normal second function. Then add to this the differentiated second function multiplied by the normal first function.

If $y = uv$, then

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

The Quotient Rule

You can differentiate rational function in the form $\frac{u(x)}{v(x)}$ by using the quotient rule. It is used when one function has been divided by another, for example $\frac{x}{2x+5}$

If $y = \frac{u}{v}$, then

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Standard Derivatives

There are certain functions that have standard derivatives, some of which are given in the formula sheet, such as tan, sec, cot, and cosec. Others need to be remembered, such as ln(x), e^x, sin x and cos x.

- If $y = e^{f(x)}$, then $\frac{dy}{dx} = f'(x)\, e^{f(x)}$

One of the most important features of the function $f(x) = e^x$ is that this function is its own derivative, i.e. if $f(x) = e^x$, then $f'(x) = e^x$ also. This result can be used with the chain, product and quotient rules to enable you to differentiate other functions, giving te standard result above.

- If $y = \ln[\mathrm{f}(x)]$, then $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$

This is a more general version of ln(x) differentiating to 1/x, and is found by using the chain rule.

- If $y = \sin x$, then $\frac{dy}{dx} = \cos x$

This result can be used along with the chain rule to differentiate more complicated expressions, or along with the quotient rule and the derivative of cos x to work out the derivative of tan x.

- If $y = \cos x$, then $\frac{dy}{dx} = -\sin x$

This result can be used along with the chain rule to differentiate more complicated expressions, or along with the quotient rule and the derivative of sin x to work out the derivative of tan x.

The formula that are given in the formula sheet:

- If $y = \tan x$, then $\frac{dy}{dx} = \sec^2 x$
- If $y = \operatorname{cosec} x$, then $\frac{dy}{dx} = -\operatorname{cosec} x \cot x$
- If $y = \sec x$, then $\frac{dy}{dx} = \sec x \tan x$
- If $y = \cot x$, then $\frac{dy}{dx} = -\operatorname{cosec}^2 x$