CORE 3 SUMMARY NOTES

1 Functions

- A function is a rule which generates exactly ONE OUTPUT for EVERY INPUT.
  To be defined fully the function has
  a RULE – tells you how to calculate the output from the input
  a DOMAIN – the set of values which will be used as inputs

  e.g. $f(x) = \sqrt{x}$ *domain* $x \geq 0$
  (cannot find the square root of negative values)

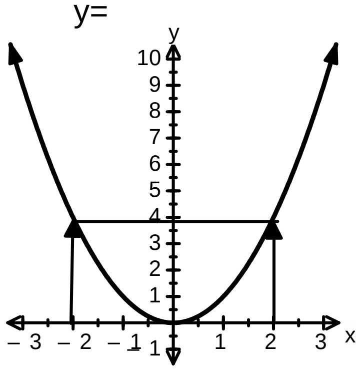
- ALTERNATIVE NOTATION
  $f : x \mapsto x^2$ means function f such that x maps to x^2
  input x is converted to output x^2

  $x \in \mathbb{R}$ x can be any real number

- There are different types of functions

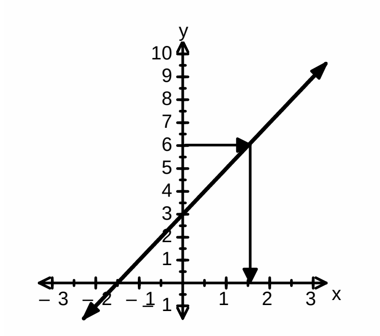
MANY-ONE

y=



$y = x^2$
2 different inputs give the same output

ONE-ONE



$y = 2x + 3$
for each output there is only one possible input

- The RANGE of a function is the complete set of all of the OUTPUTS

- An INVERSE function is denoted by f^{-1}.

  ONLY ONE-ONE FUNCTIONS HAVE INVERSES

  The DOMAIN of an inverse function is the RANGE of the function

  e.g. The function f is defined by $f(x) = \frac{3}{2x-1}$ *find* $f^{-1}(x)$

  Step 1 : Write the rule in terms of x and y

  $$y = \frac{3}{2x-1}$$

  Step 2 : Rearrange to make x the subject

  $$x = \frac{3+y}{2y}$$

  Step 3 : Replace the y's with x's

  $$f^{-1}(x) = \frac{3+x}{2x}$$

- Using the same scale on the x and y axis, the graphs of a function and it's inverse have **reflection symmetry** in the line **y = x**

<u>COMPOSITE FUNCTIONS</u>

The function **gf** is called a **composite** function and tells you to' do f first then gf(x)

e.g. $f(x) = 2x + 3$ $g(x) = x^2 + 2$

$gf(x) = (2x+3)^2 + 2$
$gf(x) = 4x^2 + 12x + 11$

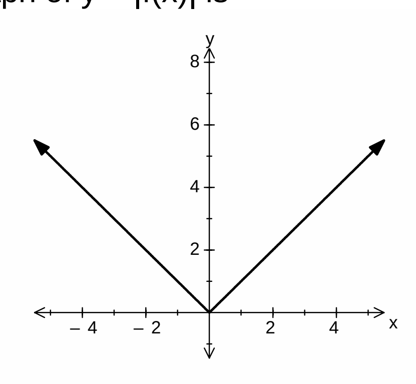
$fg(x) = 2(x^2 + 2) + 3$
$fg(x) = 2x^2 + 7$

2 The Modulus function

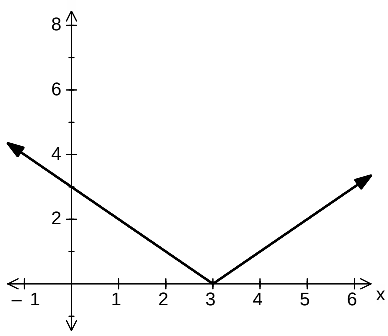
- |x| is the 'modulus of x' or the 'absolute value'
- The modulus of a real number can be thought of as its' distance' from 0 and it is always positive.

  $|4| = 4$ $|-2| = 2$

- The graph of y = |f(x)| is



$y = |x - 3|$ translation $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$

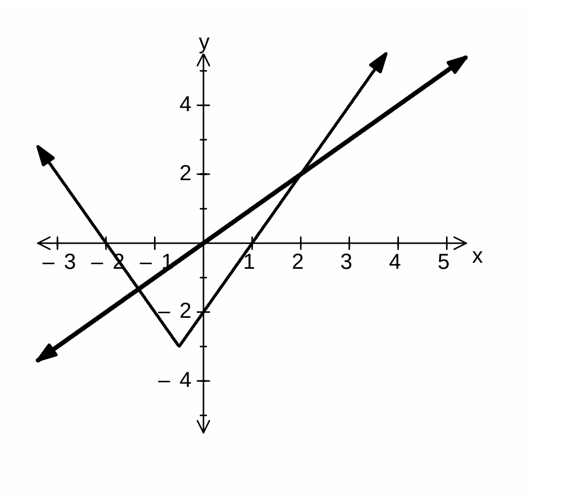


To sketch the graph of y = |(fx)| first sketch the graph of y = f(x) Take any part of the graph that is below the x-axis and reflect it in the x-axis.

<u>SOLVING EQUATIONS</u>

Always sketch the graph before you start to determine the number of solutions

A function is defined by $f(x) = |2x+1| - 3$
Solve the inequality $f(x) < x$



The graphs shows 2 solutions

$(2x+1) - 3 = x$ $-(2x + 1) - 3 = x$

$2x - 2 = x$ $-2x - 4 = x$

$x = 2$ *or* $x = -\frac{4}{3}$

3 Transforming Graphs

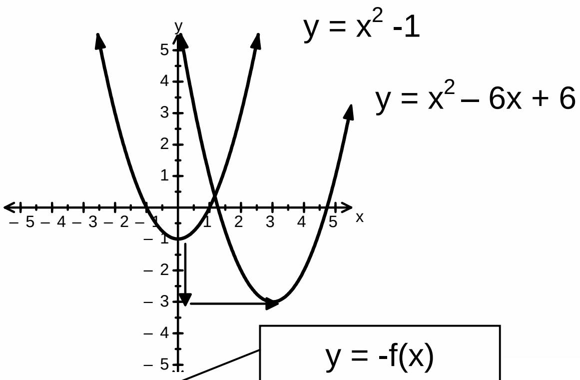
- TRANSLATION - to find the equation of a graph after a translation of $\begin{bmatrix} a \\ b \end{bmatrix}$ you replace x by (x-a) and y by (y - b)

e.g. The graph of $y = x^2 - 1$ is translated through $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$. Write down the equation of the graph formed.

$(y + 2) = (x-3)^2 - 1$

$y = x^2 - 6x + 6$

$y - b = f(x-a)$
or
$y = f(x-a) + b$



- REFLECTING

Reflection in the x-axis, replace y with $-y$ — $y = -f(x)$

Reflection in the y-axis, replace x with $-x$ — $y = f(-x)$

- STRETCHING

Stretch of factor k in the x direction replace x by $\frac{1}{k}x$ — $y = f(\frac{1}{k}x)$

Stretch of factor k in the y direction replace y by $\frac{1}{k}y$ — $y = kf(x)$

- COMBINING TRANSFORMATIONS

When applying 2 transformations the order does not matter if one involves replacing x and the other replacing y. If both transformations involve replacing x (or y) then the order could matter

e.g. The graph of $y = x^2$ is first translated by $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$ and then reflected in the y-axis

Find the equation of the final image.

Translation $y = (x - 3)^2$

Reflection $y = (-x - 3)^2$

$y = (x + 3)^2$

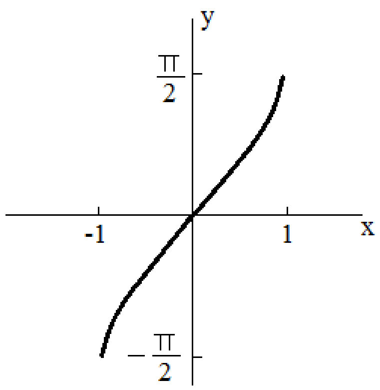
4 TRIGONOMETRY

INVERSE FUNCTIONS

$y = \sin^{-1} x$ arcsin x or asin x

domain $-1 \le x \le 1$

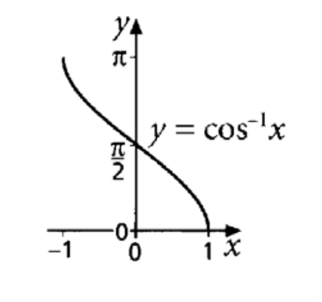
range $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$



$y = \cos^{-1} x$ arccos x or acos x

domain $-1 \le x \le 1$

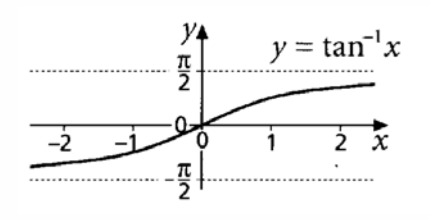
range $0 \le y \le \pi$

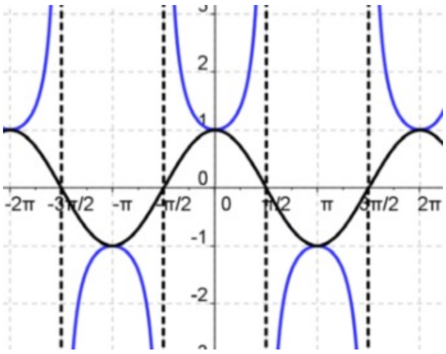


$y = \tan^{-1} x$ arctan x or atan x

domain $x \in R$

range $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$



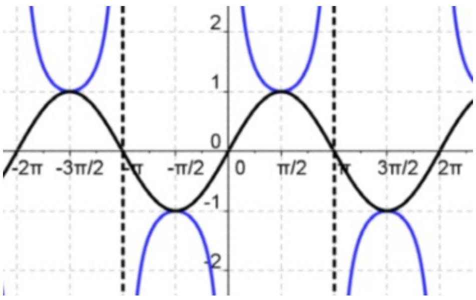


sec x is defined as $\frac{1}{\cos x}$

y = sec x has domain $x \in$

$x \in R \quad x \neq \pm\frac{\pi}{2}, \ \pm\frac{3\pi}{2}, \frac{\pm 5\pi}{2}$

and range $y \le -1$ and $y \ge 1$



cosec c is defined as $\frac{1}{\sin x}$

y = cosec x has domain

$x \in R \quad x \neq 0, \pm\pi, \pm 2\pi, \pm 3\pi$

and range $y \le -1$ and $y \ge 1$

5 Natural Logarithms and e^x

- e is an irrational; its value is 2.718281828 correct to 9 decimal places
- **Natural Logarithms** use e as a base and we write $\log_e x$ as ln x

$$e^x = y \quad \Rightarrow \quad x = \ln y$$

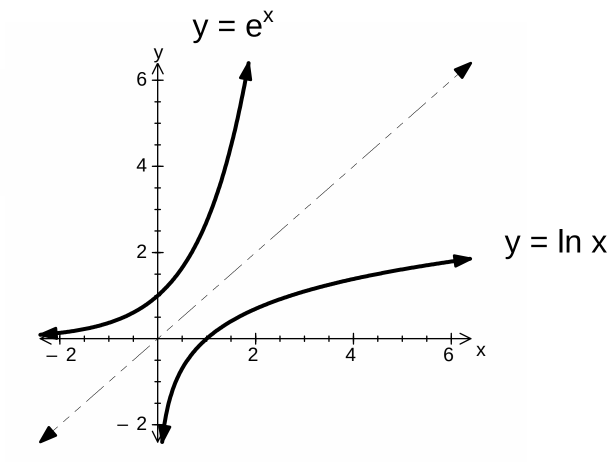
e.g. Solve the equation $e^{-5x} - 3 = 0$

$$e^{-5x} = 3$$
$$-5x = \ln 3$$
$$x = -\frac{1}{5}\ln^{3}$$
$$= -0{\cdot}2197$$

*If the question asks for an **exact** answer do not change into decimals*

- $e^x = y \Rightarrow x = \ln y$ so e^x and ln x are inverse functions

e^x is positive for all x so ln x is defined only for positive values of x



e.g The function g is defined by $g(x) = 2e^{x-5} + 3$ for all real x. Find and expression for $g^{-1}(x)$ and state it's domain and range.

$$y = 2e^{x-5} + 3$$
$$y - 3 = 2e^{x-5}$$
$$½(y - 3) = e^{x-5}$$
$$\ln(½(y - 3)) = x - 5$$
$$\ln(½(y - 3)) + 5 = x$$

$$g^{-1}(x) = \ln(½(x - 3)) + 5$$

The range of g is $g(x) > 3$ so the domain of $g^{-1}(x)$ is $x > 3$
The domain of g is all real values of x so the range $g^{-1}(x)$ is all real values

- <u>Transformation of graphs</u>

  e.g. Describe the sequence of geometrical transformations needed to obtain the graph of $y = 2e^{-x}$ from the graph of $y = e^x$.

  Reflection in the y-axis gives $y = e^{-x}$
  Stretch factor of 2 in the y direction gives $y = 2e^{-x}$

6 Differentiation

Key points from C1 and C2

- The derivative of $x^n = nx^{n-1}$
- If $f'(a) > 0$, f is increasing at x = a. If $f'(a) < 0$, f is decreasing at x =a
- The points where $f'(a) = 0$ are called stationary points
  If $f''(a) > 0$ then x = a is a local minimum
  If $f''(a) < 0$ then x = a is a local maximum

- The derivative of e^x is e^x
- The derivative of ln x is $\frac{1}{x}$

e.g. If $f(x) = e^x + \ln(2x^3)$ find $f'(x)$

$$f(x) = e^x + \ln 2 + 3\ln x$$

$$f'(x) = e^x + \frac{3}{x}$$

- **Product Rule**

If y = uv then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

- **Quotient Rule**

If y = $\frac{u}{v}$ then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

- **Chain Rule**

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Find $\frac{dy}{dx}$ given that $\ln(1 + x^2)$

Let $u = 1 + x^2$ so $y = \ln u$

$$\frac{du}{dx} = 2x \qquad \frac{dy}{du} = \frac{1}{u}$$

$$\frac{dy}{dx} = \frac{2x}{1 + x^2}$$

- **Differentiating sin x, cos x and tan x**

The derivative of **sin x** is **cos x**.

The derivative of **cos x** is **–sin x**

The derivative of **tan x** is **$\sec^2 x$**

- The derivative of f(ax) is $af'(ax)$
- The derivative of f(ax+b) is $af'(ax + b)$

e.g. Find $f'(x)$ given that $f(x) = \sin 3x \cos 2x$

Let u=sin 3x and v=cos 2x

$$\frac{du}{dx} = 3\cos 3x \qquad \frac{dv}{dx} = -2\sin 2x$$

$$\frac{dy}{dx} = (\sin 3x)(-2\sin 2x) + (\cos 2x)(3\cos 3x)$$

$$\frac{dy}{dx} = 3\cos 2x\cos 3x - 2\sin 2x\sin 3x$$

7 Integration

Key points from C1 and C2

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$$

$\int_b^a f(x)\,dx$ Gives the area under the graph of y=f(x) between x=a and x=b

Areas below the x-axis are negative

- Key integrals TO LEARN

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c \qquad \int \frac{1}{x}\,dx = \ln|x| + c \qquad \int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + c \qquad \int \cos ax\,dx = \frac{1}{a}\sin ax + c$$

- Integration by SUBSTITUTION

e.g. Use the substitution $u = 1-x^2$ to find $\int x\sqrt{1-x^2}$

First find du in terms of dx $\quad \frac{du}{dx} = -2x \quad$ so $\quad du = -2x\,dx$

Rewrite the function in terms of u and du

$$\int x\sqrt{1-x^2}\,dx = -\frac{1}{2}\int \sqrt{1-x^2}\,(-2x)dx$$

$$= -\frac{1}{2}\int \sqrt{u}\,du = -\frac{1}{2}\int u^{-\frac{1}{2}}\,du$$

Carry out the integration in terms of u

$$= -\frac{1}{2}\left(\frac{2}{3}u^{\frac{3}{2}}\right) + c = -\frac{1}{3}u^{\frac{3}{2}} + c$$

Rewrite the result in terms of x

$$-\frac{1}{3}(1-x^2)^{\frac{3}{2}} + c$$

- Integration by parts

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx + c$$

e.g. Find $\int x e^{5x}\,dx$

Let $u = x$ *and* $\frac{dv}{dx} = e^{5x}$

$\frac{du}{dx} = 1 \qquad v = \frac{1}{5}e^{5x}$

$$\int xe^{5x} = \frac{x}{5}e^{5x} - \int \frac{1}{5}e^{5x}\,dx$$

$$= \frac{x}{5}e^{5x} - \frac{1}{25}e^{5x} + c$$

- Integrating $\frac{f'(x)}{f(x)}$ (the numerator is a multiple of the derivative of the denominator)

$$\int \frac{f'(x)}{f(x)} = \ln|f(x)| + c$$

e.g. Find $\int \frac{x^3}{x^4+1}\,dx$

The derivative of the denominator, x^4+1 is $4x^3$, so think of the numerator as ¼ $(4x^3)$

$$\int \frac{x^3}{x^4+1}\,dx = \frac{1}{4}\int \frac{4x^3}{x^4+1}\,dx = \frac{1}{4}\ln|x^4+1| + c$$

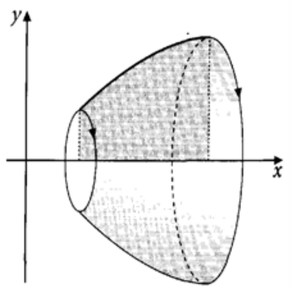
- $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$
- $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

STANDARD INTEGRALS TO LEARN

8 Solids of Revolution

- <u>Revolution about the x-axis</u>

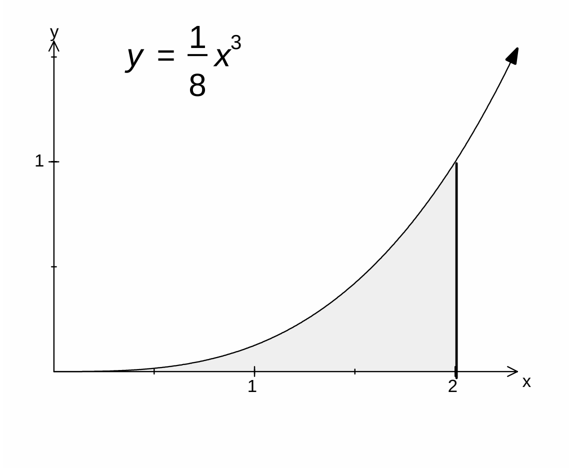
The volume of a solid of revolution about the x-axis between x = a and x = b is given by $\int_a^b \pi y^2\,dx$



- <u>Revolution about the y-axis</u>

  The volume of a solid of revolution about the y-axis between y = a and y = b is given by $\int_a^b \pi x^2 \, dy$

  e.g. The region shown is rotated through 2π radians about the y axis.
  Find the volume of the solid generated.



First express x^2 in terms of y

$$y = \frac{1}{8}x^3 \Rightarrow 8y = x^3 \Rightarrow 2y^{\frac{1}{3}} = x \Rightarrow x^2 = 4y^{\frac{2}{3}}$$

When x = 0 y = 0 when x = 2 y = 1

$$\textit{Volume} = \int_0^1 \pi x^2 \, dy = \int_0^1 4y^{\frac{2}{3}} \, dy = 4\pi \left[\frac{3y^{\frac{5}{3}}}{5} \right] = \frac{12}{5}\pi$$

9 Numerical Methods

- <u>Change of sign</u>

  For an equation f(x) =0, if $f(x_1)$ and $f(x_2)$ have opposite signs and f(x) is continuous between x_1 and x_2, then a root (solution) of the equation lies between x_1 and x_2

- <u>Staircase and Cobweb Diagrams</u>

  If an **iterative** formula (**recurrence relation)** of the form $x_{n+1} = f(x_n)$ converges to a limit, the value of the limit is the x-coordinate of the point of intersection of the graphs y=f(x) and y = x

  The limit is therefore the solution of the equation f(x) = x

  A **staircase** or **cobweb** diagram based on the graphs of y = f(x) and y = x illustrates the convergence

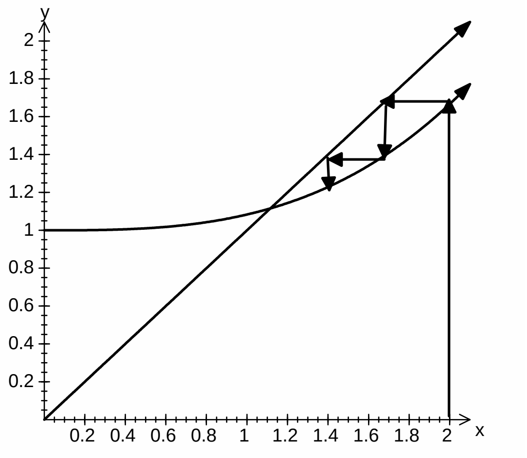
e.g Solve the equation $x^3 - 12x + 12 = 0$

First we will write it in the form x = f(x)

$$x^3 + 12 = 12x \Rightarrow \frac{x^3}{12} + 1 = x$$

Plotting the graphs $y = \frac{x^3}{12} + 1$ and $y = x$

the solution is the point of intersection of the two graphs.



We can confirm that there is a point of intersection between x = 1 and x = 2 by a change of sign the values are substituted.

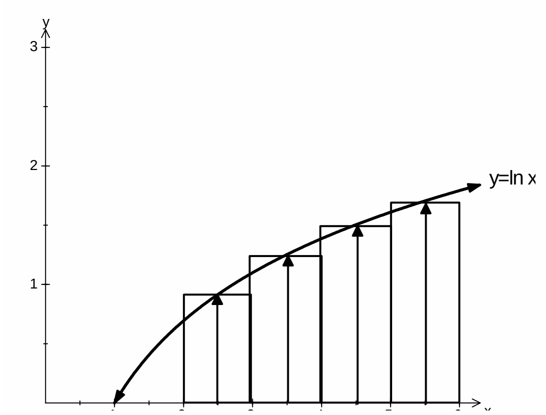
Substituting x = 2 into $y = \frac{x^3}{12} + 1$ *gives y = 1.66… (shown on the diagram)*
Substituting x = 1.66…. y = 1.38…
Repeating this the values converge to 1.1157

The solution of $x^3 - 12x + 12 = 0$ *is x = 1.1157*

- <u>The mid-ordinate Rule (Numerical Integration)</u>
  Gives and approximation to the area under a graph.
  The area is divided into strips of equal width.
  The value of the function halfway across each strip (the mid-ordinate) is calculated

  Total area = width of strip x sum of mid-ordinates

x	Y
2.5	ln 2.5
3.5	ln 3.5
4.5	ln 4.5
5.5	ln 5.5



Area = 1 x (ln 2.5 +ln 3.5 + ln 3.5 + ln 5.5)
= ln (2.5 x3.5 x 4.5 x 5.5)
= ln 216.5625
= 5.38

- <u>Simpson's Rule</u>
  Gives a more accurate approximation to the area under a graph.
  An **even** number of strips of equal width are used.
  The ordinates y_0, y_1, y_2,…. are the values of the function on the vertical edges of the strips. The area is given by

  $\frac{1}{3}$h(sum of end ordinates + 4 x sum of **odd** ordinates +
  2 x sum of remaining **even** ordinates)