Review Exercise 1

1 $\dfrac{4x}{x^2-2x-3}+\dfrac{1}{x^2+x}=\dfrac{4x}{(x-3)(x+1)}+\dfrac{1}{x(x+1)}$

$$=\frac{4x(x)+1(x-3)}{x(x+1)(x-3)}=\frac{4x^2+x-3}{x(x+1)(x-3)}$$

$$=\frac{(x+1)(4x-3)}{(x+1)x(x-3)}=\frac{4x-3}{x(x-3)}$$

2 **a** $\mathrm{f}(x)=1-\dfrac{3}{x+2}+\dfrac{3}{(x+2)^2}$

$$=\frac{(x+2)^2-3(x+2)+3}{(x+2)^2}$$

$$=\frac{x^2+4x+4-3x-6+3}{(x+2)^2}$$

$$=\frac{x^2+x+1}{(x+2)^2}$$

b $x^2+x+1=\left(x+\frac{1}{2}\right)^2+\frac{3}{4}$ ← Use the method of completing the square

$\geqslant\frac{3}{4}$ ← As $\left(x+\frac{1}{2}\right)^2\geqslant 0$

>0

for all values of x, $x\neq 2$

c $\mathrm{f}(x)=\dfrac{x^2+x+1}{(x+2)^2}$ from (a)

$\dfrac{x^2+x+1}{(x+2)^2}>0$

as $x^2+x+1>0$ from (b)

and $(x+2)^2>0$, for $x\neq -2$

So $\mathrm{f}(x)>0$, for $x\neq 2$

3 $\dfrac{3x^2+6x-2}{x^2+4}\equiv d+\dfrac{ex+f}{x^2+4}$

$\Rightarrow 3x^2+6x-2=d(x^2+4)+ex+f$

Compare coefficients of x^2: $3=d$

Compare coefficients of x: $6=e$

Compare constant terms: $-2=4d+f$

So $f=-2-4d=-2-4(3)=-14$

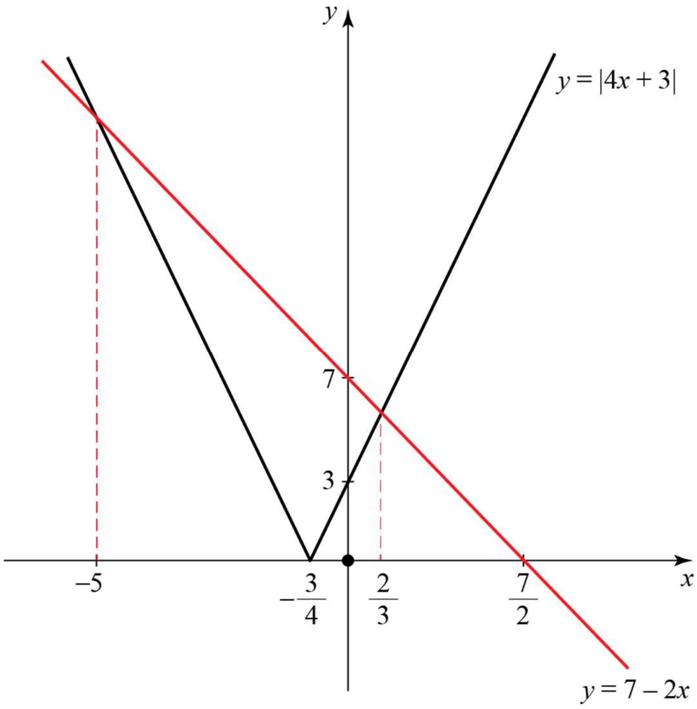
Solution: $d=3$, $e=6$, $f=-14$

4 First solve $|4x-3| = 7-2x$

$x > -\frac{3}{4}$: $4x+3 = 7-2x \Rightarrow x = \frac{2}{3}$

$x < -\frac{3}{4}$: $-(4x+3) = 7-2x \Rightarrow x = -5$

Now draw the lines $y = |4x+3|$ and $y = 7-2x$



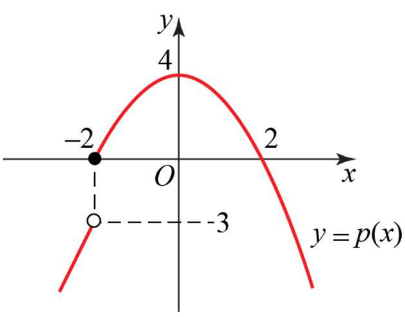
From the graph, we see that $|4x+3| > 7-2x$ when $x < -5$ or $x > \frac{2}{3}$

5 **a** For $x < -2$, p(x) is a straight line with gradient 4.

At $x = -2$, there is a discontinuity. p(–2) = 0 so draw an open dot at (–2, –3) where the line section ends and a solid dot at (–2, 0) where p(x) is defined.

For $x > -2$, $p(x) = 4 - x^2$. There is a maximum at (0, 4) since $x^2 \geqslant 0$, and the curve intersects the x-axis at (2, 0) since $4 - x^2 = 0 \Rightarrow x = \pm 2$

From the diagram, the range is $p(x) \leqslant 4$



5 b $\text{p}(a) = -20$

Check both sections of the domain for solutions.
$x < -2: \quad 4x + 5 = -20 \Rightarrow x = -\frac{25}{4}$
This is less than –2 so it is a solution.

$x \geqslant -2: \; 4 - x^2 = -20 \Rightarrow x = \pm 2\sqrt{6}$
But $-2\sqrt{6} < -2$ so discard this possibility; $a = 2\sqrt{6} \geqslant 2$ so is a solution
Solutions are $a = -\frac{25}{4}, \, a = 2\sqrt{6}$

6 a $\text{qp}(x) = 2\left(\dfrac{1}{x+4}\right) - 5 = \dfrac{2 - 5(x+4)}{x+4} = \dfrac{-5x - 18}{x+4}, \; x \neq -4$
Solutions are: $a = -5, b = -18, c = 1, d = 4$

b $\text{qp}(x) = 15$

$$\Rightarrow \frac{-5x-18}{x+4} = 15$$
$$-5x - 18 = 15(x+4) = 15x + 60$$
$$-5x - 18 = 15x + 60$$
$$20x = -78$$
$$x = -\tfrac{39}{10}$$

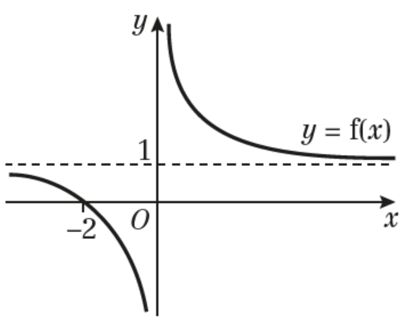
c Let $y = \text{r}(x)$

$$y = \frac{-5x-18}{x+4}$$
$$y(x+4) = -5x - 18$$
$$x(y+5) = -4y - 18$$
$$x = \frac{-4y-18}{y+5}$$

So $\text{r}^{-1}(x) = \dfrac{-4x-18}{x+5}, \; x \in \mathbb{R}, x \neq -5$

7 a $\dfrac{x+2}{x} = 1 + \dfrac{2}{x}$

Sketch $y = \dfrac{1}{x}$, stretch by a factor of 2 in the y-direction, translate by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$



7 b $f^2(x) = f\left(\dfrac{x+2}{x}\right)$

$= \dfrac{\frac{x+2}{2}+2}{\frac{x+2}{x}}$

$= \dfrac{(3x+2)}{x} \times \dfrac{x}{(x+2)}$

$= \dfrac{3x+2}{x+2}$

$\dfrac{\dfrac{x+2+2x}{x}}{\dfrac{x+2}{x}}$

c $gf\left(\dfrac{1}{4}\right) = g\left(\dfrac{2+\frac{1}{4}}{\frac{1}{4}}\right) = g(9)$

$= \ln(18-5)$

$= \ln 13$

d Let $y = \ln(2x-5)$

$e^y = 2x-5$

$\Rightarrow x = \dfrac{e^y+5}{2}$

$g^{-1}(x) = \dfrac{e^x+5}{2}, \quad x \in \mathbb{R}$

The range of $g(x)$ is $x \in \mathbb{R}$ so the domain of $g^{-1}(x)$ is $x \in \mathbb{R}$

8 a $pq(x) = 3(1-2x)+b = 3+b-6x$

$qp(x) = 1-2(3x+b) = 1-2b-6x$

As $pq(x) = qp(x)$

$\Rightarrow 3+b-6x = 1-2b-6x$

$\Rightarrow b = -\frac{2}{3}$

b Let $y = p(x)$

$y = 3x - \frac{2}{3}$

$\Rightarrow x = \dfrac{2+3y}{9}$

$p^{-1}(x) = \dfrac{3x+2}{9}$

Let $z = q(x)$

$z = 1-2x$

$\Rightarrow x = \dfrac{1-z}{2}$

$q^{-1}(x) = \dfrac{1-x}{2}$

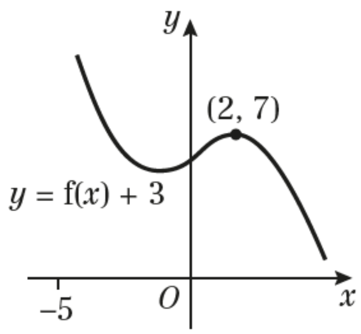
c $p^{-1}q^{-1}(x) = \dfrac{2+3\left(\dfrac{1-x}{2}\right)}{9} = \dfrac{-3x+7}{18}$

$q^{-1}p^{-1}(x) = \dfrac{1-\dfrac{2+3x}{9}}{2} = \dfrac{-3x+7}{18}$

So $p^{-1}q^{-1}(x) = q^{-1}p^{-1}(x)$

And $a = -3, b = 7, c = 18$

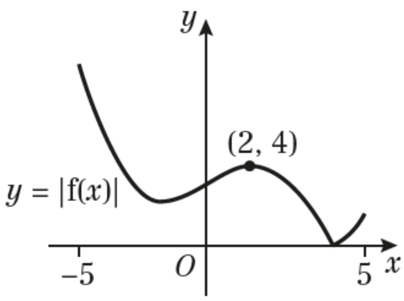
9 **a** Translation of $+3$ in the y direction. The maximum turning point is (2, 7).



b For $y \geqslant 0$, curve is $y = f(x)$

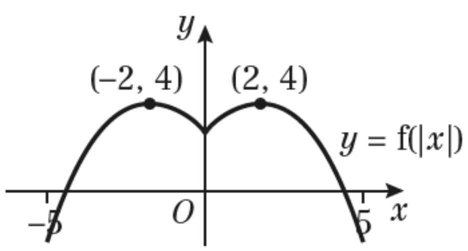
For $y < 0$, reflect in x-axis.

The maximum turning point is (2, 4)



c For $x < 0$, $f|x| = f(-x)$, so draw $y = f(x)$ for $x \geqslant 0$, and then reflect this in $x = 0$

The maximum turning points are (–2, 4) and (2, 4)



10 a To find intersections with the x-axis, solve $\text{h}(x) = 0$

$2(x+3)^2 - 8 = 0$

$\Rightarrow (x+3)^2 = 4$

$\Rightarrow x = -3 \pm 2$

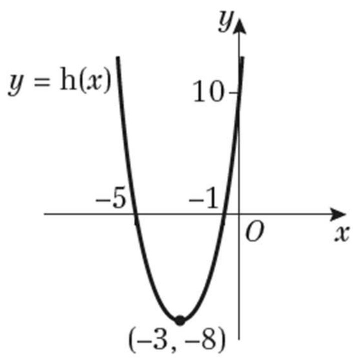
So there are intersections at (–5, 0) and (–1, 0)

To find intersections with the y-axis, find h(0)

$\text{h}(0) = 2(3)^2 - 8 = 10$

So there is an intersection at (0, 10)

Since $(x + 3)2 \geqslant 0$, there is a turning point (minimum) at (–3, –8)



b i $y = 3\text{h}(x+2)$

$\Rightarrow y = 3\left(2\left(x+2+3\right)^2 - 8\right)$

$\Rightarrow y = 6(x+5)^2 - 24$

This has a turning point when $x = -5$ at $(-5, -24)$

ii $y = \text{h}(-x)$

$\Rightarrow y = 2(-x+3)^2 - 8$

$\Rightarrow y = 2(3-x)^2 - 8$

This has a turning point when $x = 3$ at $(3, -8)$

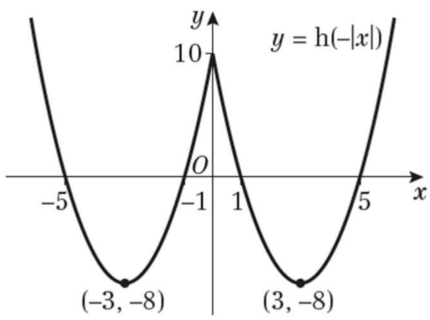
iii The modulus of $\text{h}(x)$ is the curve in part (a), with the section for $-5 < x < -1$ reflected in the x-axis. The turning point is (–3, 8)

10 c On one graph, reflect $\mathrm{h}(x)$ in the y-axis to see what $\mathrm{h}(-x)$ looks like.

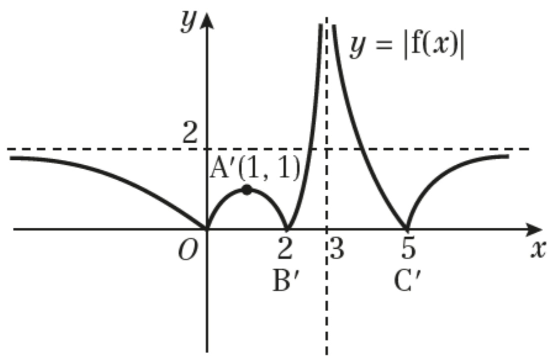
Now to obtain the sketch of $\mathrm{h}(-|x|)$, start a new graph,

copy $\mathrm{h}(-x)$ for $x \geqslant 0$, then reflect the result in the y-axis.

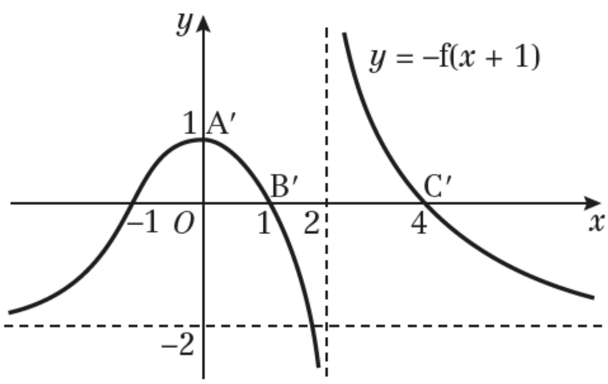
The x-intercepts are (–5, 0), (–1, 0), (1, 0), (5, 0); the y-intercept is (0, 10) and there are minimum turning points at (–3,–8) and (3,–8).



11 a i All parts of curve $y = \mathrm{f}(x)$ below the x-axis are reflected in x-axis.
$A \to (1,1)$, B and C do not move.

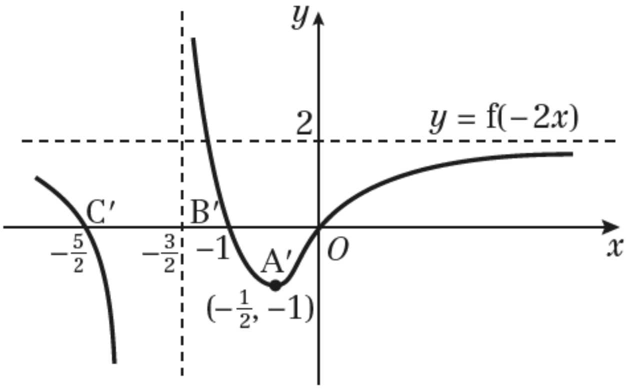


ii Translate by -1 in the x direction and reflect in the x-axis.
$A \to (0,1)$, $B \to (1,0)$, $C \to (4,0)$



11 a iii Stretch in the x direction with scale factor $\frac{1}{2}$ and reflect in the y-axis.

$A \to (-\frac{1}{2}, -1)$, $B \to (-1, 0)$, $C \to (-\frac{5}{2}, 0)$



b i $3|\text{f}(x)| = 2 \Rightarrow |\text{f}(x)| = \frac{2}{3}$

Number of solutions is 6

ii $2|\text{f}(x)| = 3 \Rightarrow |\text{f}(x)| = \frac{3}{2}$

Number of solutions is 4

> Consider graph **a i**
>
> **i** How many times does the line $y = \frac{2}{3}$ cross the curve?
> Line is below A'
>
> **ii** Draw the line $y = \frac{3}{2}$

12 a $\text{q}(x) = \frac{1}{2}|x + b| - 3$

$\text{q}(0) = \frac{|b|}{2} - 3 = \frac{3}{2} \Rightarrow |b| = 9$

$b < 0$ so $b = -9$

b A is $(9, -3)$

To find B:

$x > 9$ so solve $\frac{1}{2}(x - 9) - 3 = 0$

$\Rightarrow x = 15$

So B is (15, 0)

c $\text{q}(x) = \frac{1}{2}|x - 9| - 3 = -\frac{x}{3} + 5$

$x < 9$: $\frac{9 - x}{2} - 3 = -\frac{x}{3} + 5$

$3(9 - x) - 18 = -2x + 30$

$27 - 18 - 30 = x$

$x = -21$

$x > 9$: $\frac{x - 9}{2} - 3 = -\frac{x}{3} + 5$

$3(x - 9) - 18 = -2x + 30$

$5x = 27 + 18 + 30$

$5x = 75$

$x = 15$

Solution set; −21, 15

13 a $-\frac{5}{3}|x+4| \leqslant 0 \Rightarrow$ range is $\mathrm{f}(x) \leqslant 8$

b Over the whole domain, f(x) is not a one-one function so it cannot have an inverse.

c First solve $-\frac{5}{3}|x+4|+8=\frac{2}{3}x+4$

$x<4$: $\frac{5}{3}(x+4)+8=\frac{2}{3}x+4$

$5(x+4)+24=2x+12$

$3x=12-24-20$

$x=-\frac{32}{3}$

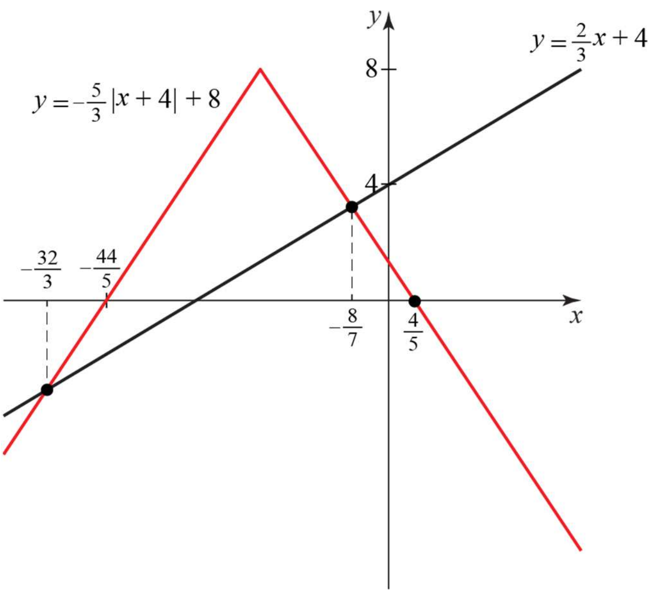
$x>4$: $-\frac{5}{3}(x+4)+8=\frac{2}{3}x+4$

$-5(x+4)+24=2x+12$

$7x=-20+24-12$

$x=-\frac{8}{7}$

Now sketch the lines $y=-\frac{5}{3}|x+4|+8$ and $y=\frac{2}{3}x+4$



From the graph we see that the inequality is satisfied in the region

$-\frac{32}{3}<x<-\frac{8}{7}$

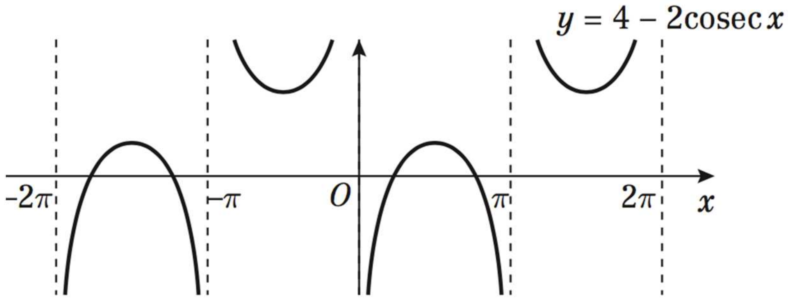
d From the sketch drawn from part (c), the equation will have no solutions if the line lies above the apex of f(x) at (–4, 8)

$\Rightarrow \frac{5}{3}(-4)+k>8$

$\Rightarrow k>8+\frac{20}{3}$

$\Rightarrow k>\frac{44}{3}$

14 a $y = 4 - 2\text{cosec}\,x$ is $y = \text{cosec}\,x$ stretched by a scale factor 2 in the y-direction, then reflected in the x-axis and then translated by the vector $\begin{pmatrix}0\\4\end{pmatrix}$



b The minima in the graph occur when $\text{cosec}\,x = -1$ and $y = 6$. The maxima occur when $\text{cosec}\,x = 1$ and $y = 2$. So there are no solutions for $2 < k < 6$.

15 a The graph is a translation of $y = \sec\theta$ by α.

So $\alpha = \frac{\pi}{3}$

b As the curve passes through (0, 4)

$$4 = k\sec\frac{\pi}{3} \Rightarrow k = 4\cos\frac{\pi}{3} = 2$$

c $-2\sqrt{2} = 2\sec\left(\theta - \frac{\pi}{3}\right)$

$$\Rightarrow \cos\left(\theta - \frac{\pi}{3}\right) = -\frac{1}{\sqrt{2}}$$

$$\Rightarrow \theta - \frac{\pi}{3} = -\frac{5\pi}{4}, -\frac{3\pi}{4}$$

$$\Rightarrow \theta = -\frac{11\pi}{12}, -\frac{5\pi}{12}$$

16 a
$$\frac{\cos x}{1-\sin x} + \frac{1-\sin x}{\cos x} \equiv \frac{\cos^2 x + (1-\sin x)^2}{\cos x(1-\sin x)}$$

$$\equiv \frac{\cos^2 x + 1 - 2\sin x + \sin^2 x}{\cos x(1-\sin x)}$$

$$\equiv \frac{2 - 2\sin x}{\cos x(1-\sin x)}$$

$$\equiv \frac{2}{\cos x}$$

$$\equiv 2\sec x$$

16 b By part a the equation becomes

$$2\sec x = -2\sqrt{2}$$
$$\Rightarrow \sec x = -\sqrt{2}$$
$$\Rightarrow \cos x = -\frac{1}{\sqrt{2}}$$
$$x = \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{11\pi}{4}, \frac{13\pi}{4}$$

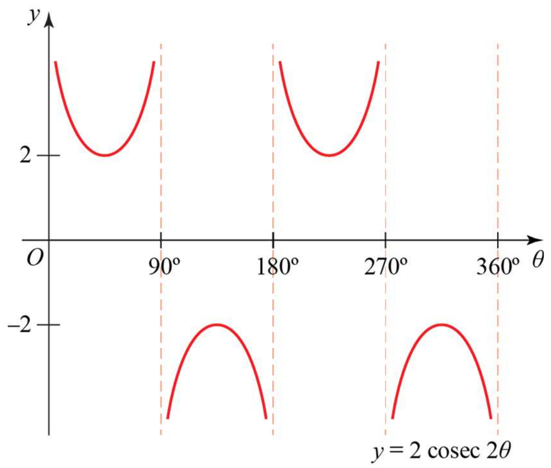
17 a $\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta}$

$= \frac{1}{\sin\theta\cos\theta}$ (using $\cos^2\theta + \sin^2\theta \equiv 1$)

$= \frac{1}{\frac{1}{2}\sin 2\theta}$ (using double-angle formula $\sin 2\theta \equiv 2\sin\theta\cos\theta$)

$= 2\operatorname{cosec} 2\theta$

b The graph of $y = 2\operatorname{cosec} 2\theta$ is a stretch of the graph of $y = \operatorname{cosec}\theta$ by a scale factor of $\frac{1}{2}$ in the horizontal direction and then a stretch by a factor of 2 in the vertical direction.



c By part a the equation becomes

$2\operatorname{cosec} 2\theta = 3$

$\Rightarrow \operatorname{cosec} 2\theta = \frac{3}{2}$

$\Rightarrow \sin 2\theta = \frac{2}{3}$, in the interval $0 \leqslant 2\theta \leqslant 720°$

Calculator value is $2\theta = 41.81°$ (2 d.p.)

Solutions are $2\theta = 41.81°, 180° - 41.81°, 360° + 41.81°, 540° - 41.81°$

So the solution set is: 20.9°, 69.1°, 200.9°, 249.1°

18 a Note the angle $BDC = \theta$

$$\cos\theta = \frac{BC}{10} \Rightarrow BC = 10\cos\theta$$

$$\sin\theta = \frac{BC}{BD} \Rightarrow BD = \frac{BC}{\sin\theta} = \frac{10\cos\theta}{\sin\theta} = 10\cot\theta$$

b $10\cot\theta = \frac{10}{\sqrt{3}}$

$\Rightarrow \cot\theta = \frac{1}{\sqrt{3}},\ \theta = \frac{\pi}{3}$

From the triangle BCD, $\cos\theta = \frac{DC}{BD}$

$\Rightarrow DC = BD\cos\theta$

So $DC = 10\cot\theta\cos\theta$

$= 10\left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{2}\right)$

$= \frac{5}{\sqrt{3}}$

19 a $\sin^2\theta + \cos^2\theta \equiv 1$

$\Rightarrow \frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} \equiv \frac{1}{\cos^2\theta}$ (dividing by $\cos^2\theta$)

$\Rightarrow \tan^2\theta + 1 \equiv \sec^2\theta$

b $2\tan^2\theta + \sec\theta = 1$

$\Rightarrow 2\sec^2\theta - 2 + \sec\theta = 1$

$\Rightarrow 2\sec^2\theta + \sec\theta - 3 = 0$

$\Rightarrow (2\sec\theta + 3)(\sec\theta - 1) = 0$

$\Rightarrow \sec\theta = -\frac{3}{2}, \sec\theta = 1$

$\Rightarrow \cos\theta = -\frac{2}{3}, \cos\theta = 1$

Solutions are $131.8°, 360° - 131.8°, 0°$

So solution set is: $0.0°, 131.8°, 228.2°$ (1 d.p.)

20 a $a = \frac{1}{\sin x} = \frac{1}{\frac{1}{2}b} = \frac{2}{b}$

20 b $\dfrac{4-b^2}{a^2-1}=\dfrac{4-b^2}{\left(\frac{2}{b}\right)^2-1}$

$=\dfrac{4-b^2}{\frac{4}{b^2}-1}=\dfrac{4-b^2}{\frac{4-b^2}{b^2}}=(4-b^2)\times\dfrac{b^2}{4-b^2}$

$=b^2$

An alternative approach is to first substitute the trigonometric functions for a and b

$\dfrac{4-b^2}{a^2-1}=\dfrac{4-4\sin^2 x}{\operatorname{cosec}^2 x-1}$

$=\dfrac{4(1-\sin^2 x)}{\cot^2 x}$

$=\dfrac{4\cos^2 x}{\cot^2 x}$

$=4\sin^2 x=b^2$

21 a $y=\arcsin x$

$\Rightarrow \sin y=x$

$x=\cos\left(\frac{\pi}{2}-y\right)$ ← Using $\sin\theta=\cos\left(\frac{\pi}{2}-\theta\right)$

$\Rightarrow \frac{\pi}{2}-y=\arccos x$

b $\arcsin x+\arccos x=y+\frac{\pi}{2}-y$

$=\frac{\pi}{2}$

22 a $\arccos\dfrac{1}{x}=p\Rightarrow\cos p=\dfrac{1}{x}$

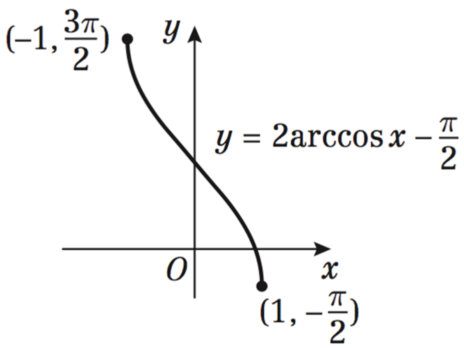
Use Pythagoras' theorem to show that opposite side of the right-angle triangle with angle p is

$\sqrt{x^2-1}$

So $\sin p=\dfrac{\sqrt{x^2-1}}{x}\Rightarrow p=\arcsin\dfrac{\sqrt{x^2-1}}{x}$

b If $0\leqslant x<1$ then x^2-1 is negative and you cannot take the square root of a negative number.

23 a $y = 2\arccos x - \frac{\pi}{2}$ is $y = \arccos x$ stretched by a scale factor of 2 in the y-direction and then translated by $-\frac{\pi}{2}$ in the vertical direction



b $2\arccos x - \frac{\pi}{2} = 0$

$\Rightarrow \arccos x = \frac{\pi}{4}$

$\Rightarrow x = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

Coordinates are $\left(\frac{1}{\sqrt{2}}, 0\right)$

24 $\tan\left(x + \frac{\pi}{6}\right) = \frac{1}{6} \Rightarrow \frac{\tan x + \frac{\sqrt{3}}{3}}{1 - \frac{\sqrt{3}}{3}\tan x} = \frac{1}{6}$ [using the addition formula for tan $(A + B)$]

$6\tan x + 2\sqrt{3} = 1 - \frac{\sqrt{3}}{3}\tan x$

$\left(\frac{18 + \sqrt{3}}{3}\right)\tan x = 1 - 2\sqrt{3}$

$\tan x = \frac{3\left(1 - 2\sqrt{3}\right)\left(18 - \sqrt{3}\right)}{\left(18 + \sqrt{3}\right)\left(18 - \sqrt{3}\right)}$

$= \frac{72 - 111\sqrt{3}}{321}$

25 a $\sin(x+30°)=2\sin(x-60°)$

So $\sin x\cos 30° + \cos x \sin 30° = 2\left(\sin x \cos 60° - \cos x \sin 60°\right)$ (using the addition formulae for sin)

$\frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x = 2\left(\frac{1}{2}\sin x - \frac{\sqrt{3}}{2}\cos x\right)$

$\sqrt{3}\sin x + \cos x = 2\sin x - 2\sqrt{3}\cos x$ (multiplying both sides by 2)

$\left(-2+\sqrt{3}\right)\sin x = \left(-1-2\sqrt{3}\right)\cos x$

So $\tan x = \frac{-1-2\sqrt{3}}{-2+\sqrt{3}}$

$= \frac{\left(-1-2\sqrt{3}\right)\left(-2-\sqrt{3}\right)}{\left(-2+\sqrt{3}\right)\left(-2-\sqrt{3}\right)}$

$= \frac{2+6+4\sqrt{3}+\sqrt{3}}{4-3}$

$= 8+5\sqrt{3}$

b $\tan(x+60°) = \frac{\tan x + \tan 60}{1-\tan x \tan 60}$

$= \frac{8+5\sqrt{3}+\sqrt{3}}{1-\left(8+5\sqrt{3}\right)\sqrt{3}}$

$= \frac{8+6\sqrt{3}}{-14-8\sqrt{3}}$

$= \frac{\left(4+3\sqrt{3}\right)\left(-7+4\sqrt{3}\right)}{\left(-7-4\sqrt{3}\right)\left(-7+4\sqrt{3}\right)}$

$= \frac{36-28-21\sqrt{3}+16\sqrt{3}}{49-48}$

$= 8-5\sqrt{3}$

26 a $\sin 165° = \sin(120°+45°)$

$= \sin 120° \cos 45° + \cos 120° \sin 45°$

$= \frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}} + \frac{-1}{2}\times\frac{1}{\sqrt{2}}$

$= \frac{\sqrt{3}-1}{2\sqrt{2}}$

$= \frac{\sqrt{6}-\sqrt{2}}{4}$

26 b $\text{cosec}\,165° = \dfrac{1}{\sin 165°}$

$= \dfrac{4}{(\sqrt{6}-\sqrt{2})} \times \dfrac{(\sqrt{6}+\sqrt{2})}{(\sqrt{6}+\sqrt{2})}$

$= \dfrac{4(\sqrt{6}+\sqrt{2})}{6-2}$

$= \sqrt{6}+\sqrt{2}$

27 a $\cos A = \frac{3}{4}$

Using Pythagoras' theorem and noting that $\sin A$ is negative as A is in the fourth quadrant, this gives

$\sin A = -\dfrac{\sqrt{7}}{4}$

Using the double-angle formula for sin gives

$\sin 2A = 2\sin A \cos A = 2\left(-\dfrac{\sqrt{7}}{4}\right)\left(\dfrac{3}{4}\right) = -\dfrac{3\sqrt{7}}{8}$

b $\cos 2A = 2\cos^2 A - 1 = \frac{1}{8}$

$\Rightarrow \tan 2A = \dfrac{\sin 2A}{\cos 2A} = \dfrac{\left(-\frac{3\sqrt{7}}{8}\right)}{\left(\frac{1}{8}\right)} = -3\sqrt{7}$

28 a $\cos 2x + \sin x = 1$

$\Rightarrow 1 - 2\sin^2 x + \sin x = 1$ (using double-angle formula for $\cos 2x$)

$\Rightarrow 2\sin^2 x - \sin x = 0$

$\Rightarrow \sin x(2\sin x - 1) = 0$

$\Rightarrow \sin x = 0, \sin x = \frac{1}{2}$

Solutions in the given interval are: $-180°, 0°, 30°, 150°, 180°$

b $\sin x(\cos x + \text{cosec}\, x) = 2\cos^2 x$

$\Rightarrow \sin x \cos x + 1 = 2\cos^2 x$

$\Rightarrow \text{in}\, x \cos x = 2\cos^2 x - 1$

$\Rightarrow \frac{1}{2}\sin 2x = \cos 2x$ (using the double-angle formulae for $\sin 2x$ and $\cos 2x$)

$\Rightarrow \tan 2x = 2$, for $-360° \leqslant 2x \leqslant 360°$

So $2x = 63.43° - 360°, 63.43° - 180°, 63.43°, 63.43° + 180°$

Solution set: $-148.3°, -58.3°, 31.7°, 121.7°$ (1 d.p.)

29 a $R\sin(x+\alpha) = R\sin x\cos\alpha + R\cos x\sin\alpha$

So $R\cos\alpha = 3$, $R\sin\alpha = 2$

$R^2\cos^2\alpha + R^2\sin^2\alpha = 3^2 + 2^2 = 9 + 4 = 13$

$\Rightarrow R = \sqrt{13}$ (as $\cos^2\alpha + \sin^2\alpha \equiv 1$)

$\tan\alpha = \frac{2}{3} \Rightarrow \alpha = 0.588$ (3 d.p.)

b $R^4 = \left(\sqrt{13}\right)^4 = 169$ since the maximum value the sin function can take is 1

c $\sqrt{13}\sin(x+0.588) = 1$

$\sin(x+0.5880) = \dfrac{1}{\sqrt{13}} = 0.27735\ldots$

$x + 0.588 = \pi - 0.281, 2\pi + 0.281$

$x = 2.273, 5.976$

30 a LHS $\equiv \cot\theta - \tan\theta$

$\equiv \dfrac{\cos\theta}{\sin\theta} - \dfrac{\sin\theta}{\cos\theta}$

$\equiv \dfrac{\cos^2\theta - \sin^2\theta}{\sin\theta\cos\theta}$

$\equiv \dfrac{\cos 2\theta}{\frac{1}{2}\sin 2\theta}$ (using the double angle formulae for $\sin 2\theta$ and $\cos 2\theta$)

$\equiv 2\cot 2\theta \equiv$ RHS

b $2\cot 2\theta = 5 \Rightarrow \cot 2\theta = \frac{5}{2} \Rightarrow \tan 2\theta = \frac{2}{5}$, for $-2\pi < 2\theta < 2\pi$

So $2\theta = 0.3805 - 2\pi, 0.3805 - \pi, 0.3805, 0.3805 + \pi$

Solution set: $-2.95, -1.38, 0.190, 1.76$

31 a LHS $\equiv \cos 3\theta$

$\equiv \cos(2\theta + \theta)$

$\equiv \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$

$\equiv (\cos^2\theta - \sin^2\theta)\cos\theta - \left(2\sin\theta\cos\theta\right)\sin\theta$

$\equiv \cos^3\theta - 3\sin^2\theta\cos\theta$

$\equiv \cos^3\theta - 3(1-\cos^2\theta)\cos\theta$

$\equiv 4\cos^3\theta - 3\cos\theta \equiv$ RHS

b From part a $\cos 3\theta = 4\dfrac{2\sqrt{2}}{27} - \sqrt{2} = -\dfrac{19\sqrt{2}}{27}$

So $\sec 3\theta = -\dfrac{27}{19\sqrt{2}} = -\dfrac{27\sqrt{2}}{38}$

32 $\sin^4\theta = (\sin^2\theta)(\sin^2\theta)$

Use the double-angle formula to write $\sin^2\theta$ in terms of $\cos 2\theta$

$\cos 2\theta = 1 - \sin^2\theta \Rightarrow \sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

Now substitute the expression for $\sin^2\theta$ and expand the brackets

So $\sin^4\theta = \left(\dfrac{1-\cos 2\theta}{2}\right)\left(\dfrac{1-\cos 2\theta}{2}\right)$

$= \frac{1}{4}(1 - 2\cos 2\theta + \cos^2 2\theta)$

Again use the double-angle formula to write $\cos^2 2\theta$ in terms of $\cos 4\theta$

So $\sin^4\theta = \dfrac{1}{4}\left(1 - 2\cos 2\theta + \dfrac{1+\cos 4\theta}{2}\right)$

$= \frac{3}{8} - \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta$

Challenge

1 a B is located where $g(x) = -\frac{3}{4}x + \frac{3}{2} = 0 \Rightarrow x = 2$

So B has coordinates $(2,0)$

To find A solve $f(x) = g(x)$ for $x < -3$

$3(x+3) + 15 = -\frac{3}{4}x + \frac{3}{2}$

$\Rightarrow 12x + 96 = -3x + 6$

$\Rightarrow 15x = -90$

$\Rightarrow x = -6$

$g(-6) = f(-6) = 6$

So A has coordinates $(-6, 6)$

M is the midpoint of A and so has coordinates $\left(\dfrac{-6+2}{2}, \dfrac{6+0}{2}\right) = (-2, 3)$

To find the radius of the circle, use Pythagoras' theorem to find the length of MA:

$|MA| = \sqrt{(2-(-2))^2 + (3-0)^2} = \sqrt{25} = 5$

Therefore the equation of the circle is

$(x+2)^2 + (y-3)^2 = 25$

Challenge

1 b For $x < -3$, $\mathrm{f}(x) = 3(x+3)+15 = 3x+24$

Substituting $y = 3x+24$ into the equation of the circle

$(x+2)^2 + (3x+21)^2 = (x+2)^2 + 9(x+7)^2 = 25$

$\Rightarrow 10x^2 + 130x + 420 = 0$

$\Rightarrow x^2 + 13x + 42 = 0$

$\Rightarrow (x+7)(x+6) = 0$

Solutions $x = -7, x = -7$

From the diagram, at P $x = -7$, and $\mathrm{f}(x) = -12+15 = 3$

So P has coordinates $(-7, 3)$

Angle $\angle APB = 90^o$ by circle theorems so the area of the triangle is $\frac{1}{2}|AP||PB|$

$|AP| = \sqrt{1^2+3^2} = \sqrt{10}$

$|PB| = \sqrt{9^2+3^2} = \sqrt{90} = 3\sqrt{10}$

$\text{Area} = \frac{1}{2}(\sqrt{10})(3\sqrt{10}) = 15$

2 $\mathrm{p}(x) = |x^2 - 8x + 12| = |(x-6)(x-2)|$

$\mathrm{q}(x) = |x^2 - 11x + 28| = |(x-4)(x-7)|$

To find the x-coordinate of A solve

$-x^2 + 8x - 12 = x^2 - 11x + 28$

$\Rightarrow 2x^2 - 19x + 40 = 0$

$$\Rightarrow x = \frac{19 - \sqrt{361 - 4(2)(40)}}{2(2)} = \frac{19 - \sqrt{41}}{4}$$

Using the quadratic formula, and from the graph we know to take the negative square root.

To find the x-coordinate of B solve

$-x^2 + 8x - 12 = -x^2 + 11x - 28$

$\Rightarrow x = \frac{16}{3}$

To find the x-coordinate of C solve

$x^2 - 8x + 12 = -x^2 + 11x - 28$

$\Rightarrow 2x^2 - 19x + 40 = 0$

$$\Rightarrow x = \frac{19 + \sqrt{41}}{4}$$

Taking the positive square root this time.

Solution is $A: \dfrac{19 - \sqrt{41}}{4}, B: \dfrac{16}{3}, C: \dfrac{19 + \sqrt{41}}{4}$

Challenge

3 **a** $\sin x$

b $\cos x$

c $\angle COA = \frac{\pi}{2} - x \Rightarrow \angle CAO = x$

$OA = 1 \div \sin x = \text{cosec } x$

d $AC = 1 \div \tan x = \cot x$

e $\tan x$

f $OB = 1 \div \cos x = \sec x$