Practice paper

1 $\frac{x^2-9}{x^2-3x} \div \frac{2x^2+5x-3}{x^2+7x} = \frac{x^2-9}{x^2-3x} \times \frac{x^2+7x}{2x^2+5x-3}$

$= \frac{(x+3)(x-3)}{x(x-3)} \times \frac{x(x+7)}{(2x-1)(x+3)}$

$= \frac{x+7}{2x-1}$

2 a $200 < V \leq 2000$

b $V = 800e^{-0.2t} + 1000e^{-0.1t} + 200$

$\frac{dV}{dt} = -160e^{-0.2t} - 100e^{-0.1t}$

When $t = 15$

$\frac{dV}{dt} = -160e^{-0.2(15)} - 100e^{-0.1(15)}$

$= -30.27...$

Therefore it is decreasing at the rate of 30 euros per year.

c When $V = 1400$

$800e^{-0.2t} + 1000e^{-0.1t} + 200 = 1400$

$800e^{-0.2t} + 1000e^{-0.1t} = 1200$

$4e^{-0.2t} + 5e^{-0.1t} = 6$

Let $x = e^{-0.1t}$

$4x^2 + 5x - 6 = 0$

$(4x-3)(x+2) = 0$

$x = \frac{3}{4}$ or $x = -2$

Since x is positive $x = \frac{3}{4}$

Therefore:

$e^{-0.1t} = \frac{3}{4}$

$-0.1t = \ln\left(\frac{3}{4}\right)$

$t = -10\ln\left(\frac{3}{4}\right)$

$= 10\ln\left(\frac{4}{3}\right)$

3 a $f(x)=(4-3x)e^x$

$=4e^x-3xe^x$

$f'(x)=4e^x-(3e^x+3xe^x)$

$=e^x-3xe^x$

$=e^x(1-3x)$

At the turning point $f'(x)=0$

Therefore:

$x=\frac{1}{3}$

At $x=\frac{1}{3}$

$f(x)=\left(4-3\left(\frac{1}{3}\right)\right)e^{\frac{1}{3}}$

$=3e^{\frac{1}{3}}$

Therefore, the turning point lies at $\left(\frac{1}{3},3e^{\frac{1}{3}}\right)$

b $f(x)\le 3e^{\frac{1}{3}}$

c $|f(x)|=|(4-3x)e^x|$

At $x=0$, $|f(x)|=4$

So the curve cuts the y-axis at (0, 4)

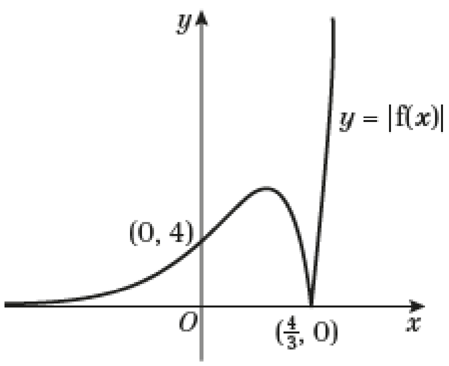
At $f(x)=0$

$|(4-3x)e^x|=0$

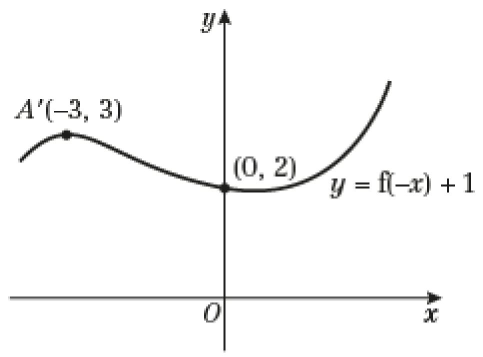
$(4-3x)e^x=0$

$x=\frac{4}{3}$

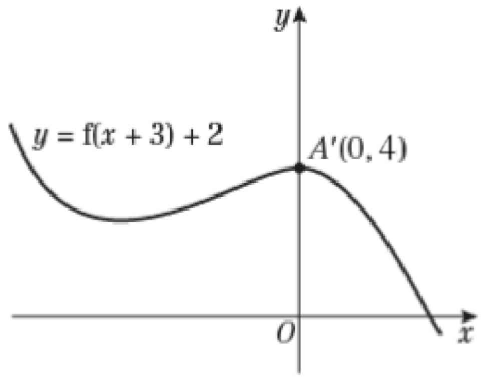
So the curve cuts the x-axis at $\left(\frac{4}{3},0\right)$



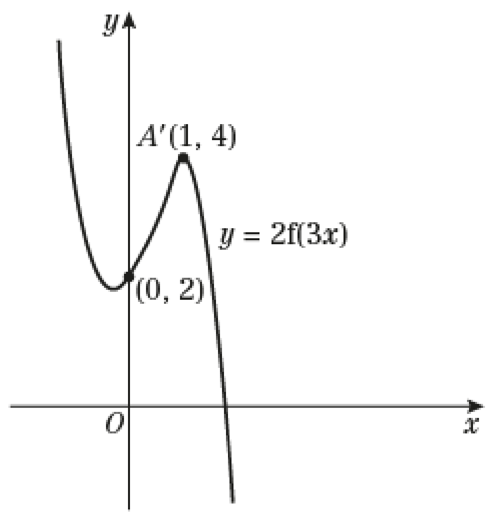
4 a



b



c



5 a $\mathrm{f}(x) = 3\sin^2 x + 2\cos^2 x$

$$= \sin^2 x + 2\sin^2 x + 2\cos^2 x$$
$$= \sin^2 x + 2(\sin^2 x + \cos^2 x)$$
$$= \sin^2 x + 2$$
$$= \frac{1-\cos 2x}{2} + 2$$
$$= \frac{1-\cos 2x + 4}{2}$$
$$= \frac{5-\cos 2x}{2} \quad \text{as required}$$

b $\int_0^{\frac{\pi}{4}} \left(\frac{5-\cos 2x}{2}\right) \mathrm{d}x = \frac{1}{2}\int_0^{\frac{\pi}{4}} (5-\cos 2x)\,\mathrm{d}x$

$$= \frac{1}{2}\left[5x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}}$$
$$= \frac{1}{2}\left(\frac{5\pi}{4} - \frac{1}{2}\sin\left(\frac{\pi}{2}\right)\right) - 0$$
$$= \frac{1}{2}\left(\frac{5\pi}{4} - \frac{1}{2}\right)$$
$$= \frac{5\pi}{8} - \frac{1}{4}$$
$$= \frac{5\pi - 2}{8}$$

6 a $y = x^2 + \sin\left(\frac{\pi}{2}x\right)$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x + \frac{\pi}{2}\cos\left(\frac{\pi}{2}x\right)$$

b At $x=-1$

$$y=(-1)^2+\sin\left(-\frac{\pi}{2}\right)$$
$$=0$$
$$\frac{dy}{dx}=2(-1)+\frac{\pi}{2}\cos\left(-\frac{\pi}{2}\right)$$
$$=-2$$

Therefore, the normal has gradient $\frac{1}{2}$

Using $y-y_1=m(x-x_1)$ with $m=\frac{1}{2}$ at $(-1, 0)$ gives:

$$y=\frac{1}{2}(x+1)$$

7 $\displaystyle\int_{\frac{\pi}{4k}}^{\frac{\pi}{3k}}(1-\pi\sin kx)\,dx=\pi\left(7-6\sqrt{2}\right)$

$$\int_{\frac{\pi}{4k}}^{\frac{\pi}{3k}}(1-\pi\sin kx)\,dx=\left[x+\frac{\pi}{k}\cos kx\right]_{\frac{\pi}{4k}}^{\frac{\pi}{3k}}$$
$$=\left[\left(\frac{\pi}{3k}+\frac{\pi}{k}\cos\left(\frac{\pi}{3}\right)\right)-\left(\frac{\pi}{4k}+\frac{\pi}{k}\cos\frac{\pi}{4}\right)\right]$$
$$=\left(\frac{\pi}{3k}-\frac{\pi}{4k}+\frac{\pi}{k}\cos\left(\frac{\pi}{3}\right)-\frac{\pi}{k}\cos\left(\frac{\pi}{4}\right)\right)$$
$$=\frac{\pi}{k}\left(\frac{1}{3}-\frac{1}{4}+\cos\left(\frac{\pi}{3}\right)-\cos\left(\frac{\pi}{4}\right)\right)$$
$$=\frac{\pi}{k}\left(\frac{1}{12}+\frac{1}{2}-\frac{\sqrt{2}}{2}\right)$$
$$=\frac{\pi}{k}\left(\frac{1}{12}+\frac{6}{12}-\frac{6\sqrt{2}}{12}\right)$$
$$=\frac{\pi}{k}\left(\frac{7-6\sqrt{2}}{12}\right)$$

$$\frac{\pi}{k}\left(\frac{7-6\sqrt{2}}{12}\right)=\pi\left(7-6\sqrt{2}\right)$$
$$\frac{1}{k}\left(\frac{1}{12}\right)=1$$
$$k=\frac{1}{12}$$

8 $\mathrm{f}(x)=\frac{3x^3-10x^2+8x+1}{x^2-4x+4}$

$=\frac{3x^3-10x^2+8x+1}{(x-2)(x-2)}$

$$\frac{3x^3-10x^2+8x+1}{(x-2)(x-2)}=Ax+B+\frac{C}{x-2}+\frac{D}{(x-2)^2}$$

$$3x^3-10x^2+8x+1=Ax(x-2)(x-2)+B(x-2)(x-2)+\frac{C(x-2)(x-2)}{x-2}+\frac{D(x-2)(x-2)}{(x-2)^2}$$

$$=Ax(x-2)(x-2)+B(x-2)(x-2)+C(x-2)+D$$

$$=A(x^3-4x^2+4x)+B(x^2-4x+4)+C(x-2)+D$$

Comparing coefficients

For x^3:
$A=3$

For x^2:
$-4A+B=-10$
$-4(3)+B=-10$
$B=2$

For x:
$4A-4B+C=8$
$4(3)-4(2)+C=8$
$12-8+C=8$
$C=4$

For constant term:
$4B-2C+D=1$
$4(2)-2(4)+D=1$
$8-8+D=1$
$D=1$

Therefore:

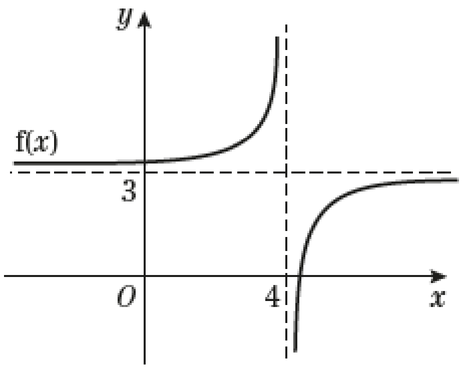
$$\frac{3x^3-10x^2+8x+1}{(x-2)(x-2)}=3x+2+\frac{4}{x-2}+\frac{1}{(x-2)^2}$$

9 a $\mathrm{f}(x) = \frac{1}{4-x} + 3$

$\mathrm{f}(3.9) = \frac{1}{4-3.9} + 3$
$= 13$

$\mathrm{f}(4.1) = \frac{1}{4-4.1} + 3$
$= -7$

b



There is a discontinuity (an asymptote) at $x = 4$ which causes the change of sign, not a root.

c At the root $\mathrm{f}(x) = 0$, therefore:

$\frac{1}{4-x} + 3 = 0$

$\frac{1+3(4-x)}{4-x} = 0$

$\frac{1+12-3x}{4-x} = 0$

$13 - 3x = 0$

$x = \frac{13}{3}$

Therefore $\alpha = \frac{13}{3}$

10 a $\int \mathrm{e}^{4x+3}\,\mathrm{d}x = \frac{1}{4}\mathrm{e}^{4x+3} + c$

10 b $\int \frac{\cos 4x}{e^{\sin 4x}} dx$

Let $u = \sin 4x$

$\frac{du}{dx} = 4\cos 4x \Rightarrow du = 4\cos 4x \, dx$

$$\int \frac{\cos 4x}{e^{\sin 4x}} dx = \frac{1}{4}\int \frac{1}{e^u} du$$

$$= \frac{1}{4}\int e^{-u} du$$

$$= -\frac{1}{4}e^{-u} + c$$

$$= -\frac{1}{4}e^{-\sin 4x} + c$$