Exercise 8A

1 a $f(x)=x^3-x+5$

$f(-2)=-8+2+5=-1<0$

$f(-1)=-1+1+5=5>0$

There is a change of sign between −2 and −1 so there is at least one root in the interval $-2<x<-1$.

b $f(x)=x^2-\sqrt{x}-10$

$f(3)=9-\sqrt{3}-10=-2.732...<0$

$f(4)=16-\sqrt{4}-10=4>0$

There is a change of sign between 3 and 4 so there is at least one root in the interval $3<x<4$.

c $f(x)=x^3-\frac{1}{x}-2$

$f(-0.5)=(-0.5)^3+2-2=-0.125<0$

$f(-0.2)=(-0.2)^3+5-2=2.992>0$

There is a change of sign between −0.5 and −0.2 so there is at least one root in the interval $-0.5<x<-0.2$.

d $f(x)=e^x-\ln x-5$

$f(1.65)=e^{1.65}-\ln 1.65-5=-0.293...<0$

$f(1.75)=e^{1.75}-\ln 1.75-5=0.194...>0$

There is a change of sign between 1.65 and 1.75 so there is at least one root in the interval $1.65<x<1.75$.

2 a $f(x)=3+x^2-x^3$

$f(1.8)=3+1.8^2-1.8^3=0.408>0$

$f(1.9)=3+1.9^2-1.9^3=-0.249<0$

There is a change of sign so there is a root, α, in the interval [1.8, 1.9].

b Choose interval [1.8635, 1.8645] to test for root.

$f(1.8635)=3+1.8635^2-1.8635^3$
$=0.00138...>0$

$f(1.8645)=3+1.8645^2-1.8645^3$
$=-0.00531...<0$

There is a change of sign between 1.8635 and 1.8645, so $1.8635<\alpha<1.8645$, which gives $\alpha=1.864$ correct to 3 d.p.

3 a $h(x)=\sqrt[3]{x}-\cos x-1$

$h(1.4)=\sqrt[3]{1.4}-\cos 1.4-1=-0.0512...<0$

$h(1.5)=\sqrt[3]{1.5}-\cos 1.5-1=0.0739...>0$

There is a change of sign so there is a root, α, in the interval [1.4, 1.5].

b Choose interval [1.4405, 1.4415] to test for root.

$h(1.4405)=\sqrt[3]{1.4405}-\cos 1.4405-1$
$=-0.00055...<0$

$h(1.4415)=\sqrt[3]{1.4415}-\cos 1.4415-1$
$=0.00069...>0$

There is a change of sign between 1.4405 and 1.4415, so $1.4405<\alpha<1.4415$, which gives $\alpha=1.441$ correct to 3 d.p.

4 a $f(x)=\sin x-\ln x$

$f(2.2)=\sin 2.2-\ln 2.2=0.020...>0$

$f(2.3)=\sin 2.3-\ln 2.3=-0.087...<0$

There is a change of sign so there is a root, α, in the interval [2.2, 2.3].

b Choose interval [2.2185, 2.2195] to test for root.

$f(2.2185)=\sin 2.2185-\ln 2.2185$
$=0.00064...>0$

$f(2.2195)=\sin 2.2195-\ln 2.2195$
$=-0.00041...<0$

There is a change of sign between 2.2185 and 2.2195, so $2.2185<\alpha<2.2195$, which gives $\alpha=2.219$ correct to 3 d.p.

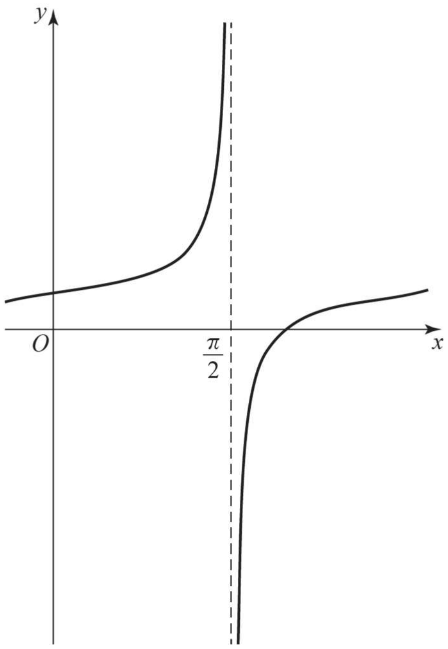
5 a $f(x)=2+\tan x$

$f(1.5)=2+\tan 1.5=16.1...>0$

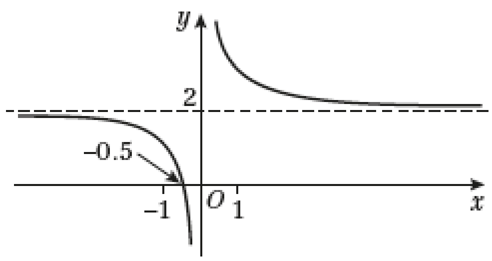
$f(1.6)=2+\tan 1.6=-32.2...<0$

So there is a change of sign in the interval [1.5, 1.6].

5 b A sketch shows there is a vertical asymptote in the graph of $y = f(x)$ at $x = \frac{\pi}{2} = 1.57...$ So there is no root in the interval [1.5, 1.6].



6 A sketch shows a root at −0.5.



Or $f(x) = 0$ when $\frac{1}{x} + 2 = 0 \Rightarrow x = -\frac{1}{2}$

which is in the interval [−1, 1].

7 a $f(x) = (105x^3 - 128x^2 + 49x - 6)\cos 2x$

$f(0.2) = (0.84 - 5.12 + 9.8 - 6)\cos 0.4$

$= -0.442... < 0$

$f(0.8) = (53.76 - 81.92 + 39.2 - 6)\cos 1.6$

$= -0.147... < 0$

b There is no sign change, so there are either no roots or an even number of roots in the interval [0.2, 0.8].

c $f(0.3) = (2.835 - 11.52 + 14.7 - 6)\cos 0.6$

$= 0.0123.. > 0$

and…

$f(0.4) = -0.111... < 0$

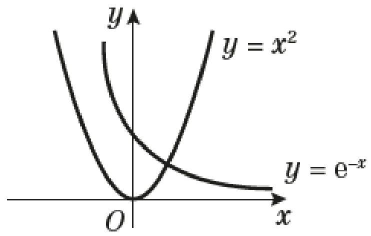
$f(0.5) = -0.202... < 0$

$f(0.6) = 0$

$f(0.7) = 0.271... > 0$

d From the changes in sign, there exists at least one root in each of the intervals $0.2 < x < 0.3$, $0.3 < x < 0.4$ and $0.7 < x < 0.8$. There is also a root at 0.6. Therefore there are at least four roots in the interval [0.2, 0.8].

8 a



b The curves meet where $e^{-x} = x^2$.
The curves meet at one point, so there is one value of x that satisfies the equation $e^{-x} = x^2$.
So $e^{-x} = x^2$ has one root.

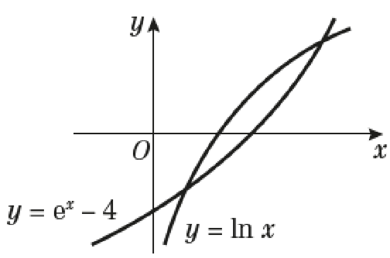
c $f(x) = e^{-x} - x^2$

$f(0.70) = e^{-0.70} - 0.70^2 = 0.0065...$

$f(0.71) = e^{-0.71} - 0.71^2 = 0.0124...$

There is a change of sign between 0.70 and 0.71 so there is at least one root in the interval $0.70 < x < 0.71$.

9 a



b The curves meet at two points, so there are two values of x that satisfy the equation $\ln x = e^x - 4$. So $\ln x = e^x - 4$ has two roots.

9 c $f(x) = \ln x - e^x + 4$

$f(1.4) = \ln 1.4 - e^{1.4} + 4 = 0.281...$

$f(1.5) = \ln 1.5 - e^{1.5} + 4 = -0.0762...$

There is a change of sign between 1.4 and 1.5 so there is at least one root in the interval $1.4 < x < 1.5$.

10 a $h(x) = \sin 2x + e^{4x}$

$h'(x) = 2\cos 2x + 4e^{4x}$

$h'(-0.9) = 2\cos(-1.8) + 4e^{-3.6}$
$= -0.345... < 0$

$h'(-0.8) = 2\cos(-1.6) + 4e^{-3.2}$
$= 0.104... > 0$

The change in sign of $h'(x)$ implies that the gradient changes from decreasing to increasing, so there is a turning point in the interval $-0.9 < x < -0.8$.

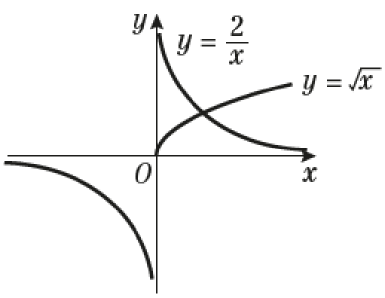
b Choose interval $[-0.8225, -0.8235]$ to test for root.

$h'(-0.8235) = 2\cos(-1.647) + 4e^{-3.294}$
$= -0.00383... < 0$

$h'(-0.8225) = 2\cos(-1.645) + 4e^{-3.29}$
$= 0.000744... > 0$

There is a change of sign between -0.8225 and -0.8235, so $-0.8225 < \alpha < -0.8235$, which gives $\alpha = -0.823$ correct to 3 d.p.

11 a



b The curves meet at one point, so there is one value of x that satisfies the equation $\sqrt{x} = \frac{2}{x}$. So $\sqrt{x} = \frac{2}{x}$ has **one** root.

c $f(x) = \sqrt{x} - \frac{2}{x}$

$f(1) = \sqrt{1} - \frac{2}{1} = -1$

$f(2) = \sqrt{2} - \frac{2}{2} = 0.414...$

There is a change of sign, so there is a root, r, between $x = 1$ and $x = 2$.

d

$\sqrt{x} = \frac{2}{x}$

$x^{\frac{1}{2}} = \frac{2}{x}$

$x^{\frac{1}{2}} \times x = 2$

$x^{\frac{1}{2}+1} = 2$

$x^{\frac{3}{2}} = 2$

$\left(x^{\frac{3}{2}}\right)^2 = 2^2$

$x^3 = 4$

So $p = 3$ and $q = 4$.

e $x^{\frac{3}{2}} = 2$

$\Rightarrow x = 2^{\frac{2}{3}} \left[= (2^2)^{\frac{1}{3}} = 4^{\frac{1}{3}}\right]$

12 a $f(x) = x^4 - 21x - 18$

$f(-0.9) = 0.6561 + 18.9 - 18 = 1.5561 > 0$

$f(-0.8) = 0.4096 + 16.8 - 18 = -0.7904 < 0$

The change of sign between -0.9 and -0.8 implies there is at least one root in the interval $[-0.9, -0.8]$.

b $f'(x) = 4x^3 - 21$

$f'(x) = 0 \Rightarrow 4x^3 = 21$

$x = \sqrt[3]{\frac{21}{4}} = 1.738...$

$f(1.738) = 1.738^4 - 21 \times 1.738 - 18$
$= -45.373...$

Stationary point is $(1.74, -45.37)$ to 2 d.p.

c $f(x) = (x-3)(x^3 + ax^2 + bx + c)$

$f(x) = x^4 + (a-3)x^3 + (b-3a)x^2 + (c-3b)x - 3c$

Comparing coefficients…

$a = 3, b = 9, c = 6$

12 d