Pure Mathematics 3 Solution Bank Ⓟ Pearson

Exercise 7B

1 a $\int \sin(2x+1)\mathrm{d}x = -\tfrac{1}{2}\cos(2x+1) + c$

b $\int 3\mathrm{e}^{2x}\mathrm{d}x = \tfrac{3}{2}\mathrm{e}^{2x} + c$

c $\int 4\mathrm{e}^{x+5}\mathrm{d}x = 4\mathrm{e}^{x+5} + c$

d $\int \cos(1-2x)\mathrm{d}x = -\tfrac{1}{2}\sin(1-2x) + c$

  OR Let $y = \sin(1-2x)$

  then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \cos(1-2x)\times(-2)$ (by chain rule)

  $\therefore \int \cos(1-2x)\mathrm{d}x = -\tfrac{1}{2}\sin(1-2x) + c$

e $\int \operatorname{cosec}^2 3x\,\mathrm{d}x = -\tfrac{1}{3}\cot 3x + c$

f $\int \sec 4x \tan 4x\,\mathrm{d}x = \tfrac{1}{4}\sec 4x + c$

g $\int 3\sin\left(\tfrac{1}{2}x+1\right)\mathrm{d}x = -6\cos\left(\tfrac{1}{2}x+1\right) + c$

h $\int \sec^2(2-x)\mathrm{d}x = -\tan(2-x) + c$

  OR Let $y = \tan(2-x)$

  then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \sec^2(2-x)\times(-1)$ (by chain rule)

  $\therefore \int \sec^2(2-x)\mathrm{d}x = -\tan(2-x) + c$

i $\int \operatorname{cosec} 2x \cot 2x\,\mathrm{d}x = -\tfrac{1}{2}\operatorname{cosec} 2x + c$

j $\int (\cos 3x - \sin 3x)\mathrm{d}x$
  $= \tfrac{1}{3}\sin 3x + \tfrac{1}{3}\cos 3x + c$
  $= \tfrac{1}{3}(\sin 3x + \cos 3x) + c$

2 a $\int (\mathrm{e}^{2x} - \tfrac{1}{2}\sin(2x-1))\mathrm{d}x$
  $= \tfrac{1}{2}\mathrm{e}^{2x} + \tfrac{1}{4}\cos(2x-1) + c$

b $\int (\mathrm{e}^x + 1)^2\,\mathrm{d}x$
  $= \int (\mathrm{e}^{2x} + 2\mathrm{e}^x + 1)\mathrm{d}x$
  $= \tfrac{1}{2}\mathrm{e}^{2x} + 2\mathrm{e}^x + x + c$

c $\int \sec^2 2x(1 + \sin 2x)\mathrm{d}x$
  $= \int (\sec^2 2x + \sec^2 2x \sin 2x)\mathrm{d}x$
  $= \int (\sec^2 2x + \sec 2x \tan 2x)\mathrm{d}x$
  $= \tfrac{1}{2}\tan 2x + \tfrac{1}{2}\sec 2x + c$

d $\int \dfrac{3 - 2\cos\left(\tfrac{1}{2}x\right)}{\sin^2\left(\tfrac{1}{2}x\right)}\mathrm{d}x$
  $= \int \left(3\operatorname{cosec}^2 \tfrac{1}{2}x - 2\operatorname{cosec}\tfrac{1}{2}x\cot\tfrac{1}{2}x\right)\mathrm{d}x$
  $= -6\cot\left(\tfrac{1}{2}x\right) + 4\operatorname{cosec}\left(\tfrac{1}{2}x\right) + c$

2 e $\int (\mathrm{e}^{3-x} + \sin(3-x) + \cos(3-x))\mathrm{d}x$
  $= -\mathrm{e}^{3-x} + \cos(3-x) - \sin(3-x) + c$

Note: extra minus signs from $-x$ terms and chain rule.

3 a $\int \dfrac{1}{2x+1}\mathrm{d}x = \tfrac{1}{2}\ln|2x+1| + c$

b $\int \dfrac{1}{(2x+1)^2}\mathrm{d}x$
  $= \int (2x+1)^{-2}\mathrm{d}x$
  $= \dfrac{(2x+1)^{-1}}{-1}\times\dfrac{1}{2} + c$
  $= -\dfrac{1}{2(2x+1)} + c$

c $\int (2x+1)^2\,\mathrm{d}x$
  $= \dfrac{(2x+1)^3}{3}\times\dfrac{1}{2} + c$
  $= \dfrac{(2x+1)^3}{6} + c$

d $\int \dfrac{3}{4x-1}\mathrm{d}x = \tfrac{3}{4}\ln|4x-1| + c$

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3 e $\int \dfrac{3}{1-4x}\,dx$

$= -\int \dfrac{3}{4x-1}\,dx$

$= -\tfrac{3}{4}\ln|4x-1| + c$

OR Let $y = \ln|1-4x|$

then $\dfrac{dy}{dx} = \dfrac{1}{1-4x} \times (-4)$ (by chain rule)

$\therefore \int \dfrac{3}{1-4x}\,dx = -\tfrac{3}{4}\ln|1-4x| + c$

Note: $\ln|1-4x| = \ln|4x-1|$ because of the absolute value sign.

f $\int \dfrac{3}{(1-4x)^2}\,dx$

$= \int 3(1-4x)^{-2}\,dx$

$= \dfrac{3}{-4} \times \dfrac{(1-4x)^{-1}}{-1} + c$

$= \dfrac{3}{4(1-4x)} + c$

g $\int (3x+2)^5\,dx = \dfrac{(3x+2)^6}{18} + c$

h $\int \dfrac{3}{(1-2x)^3}\,dx = \dfrac{3}{-2} \times \dfrac{(1-2x)^{-2}}{-2} + c$

$\qquad\qquad = \dfrac{3}{4(1-2x)^2} + c$

OR Let $y = (1-2x)^{-2}$

then $\dfrac{dy}{dx} = -2(1-2x)^{-3} \times (-2)$

(by chain rule)

$\therefore \int \dfrac{3}{(1-2x)^3}\,dx = \tfrac{3}{4}(1-2x)^{-2} + c$

4 a $\int \left(3\sin(2x+1) + \dfrac{4}{2x+1} \right) dx$

$= -\tfrac{3}{2}\cos(2x+1) + \tfrac{4}{2}\ln|2x+1| + c$

$= -\tfrac{3}{2}\cos(2x+1) + 2\ln|2x+1| + c$

b $\int \left(e^{5x} + (1-x)^5 \right) dx = \int e^{5x}\,dx + \int (1-x)^5\,dx$

$\qquad\qquad = \tfrac{1}{5}e^{5x} - \tfrac{1}{6}(1-x)^6 + c$

OR Let $y = (1-x)^6$

then $\dfrac{dy}{dx} = 6(1-x)^5 \times (-1)$ (by chain rule)

$\therefore \int (1-x)^5\,dx = -\tfrac{1}{6}(1-x)^6 + c$

c $\int \left(\dfrac{1}{\sin^2 2x} + \dfrac{1}{1+2x} + \dfrac{1}{(1+2x)^2} \right) dx$

$= \int \left(\operatorname{cosec}^2 2x + \dfrac{1}{1+2x} + (1+2x)^{-2} \right) dx$

$= -\dfrac{1}{2}\cot 2x + \dfrac{1}{2}\ln|1+2x|$

$\qquad + \dfrac{(1+2x)^{-1}}{-1} \times \dfrac{1}{2} + c$

$= -\dfrac{1}{2}\cot 2x + \dfrac{1}{2}\ln|1+2x| - \dfrac{1}{2(1+2x)} + c$

d $\int \left((3x+2)^2 + \dfrac{1}{(3x+2)^2} \right) dx$

$= \int \left((3x+2)^2 + (3x+2)^{-2} \right) dx$

$= \dfrac{(3x+2)^3}{9} - \dfrac{(3x+2)^{-1}}{3} + c$

$= \dfrac{(3x+2)^3}{9} - \dfrac{1}{3(3x+2)} + c$

5 a $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \cos(\pi - 2x)\,dx = \left[-\tfrac{1}{2}\sin(\pi - 2x) \right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}$

$= \left(-\dfrac{1}{2}\sin\left(-\dfrac{\pi}{2}\right) \right) - \left(-\dfrac{1}{2}\sin\dfrac{\pi}{2} \right)$

$= \tfrac{1}{2} + \tfrac{1}{2} = 1$

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5 b $\displaystyle\int_{\frac{1}{2}}^{1}\frac{12}{(3-2x)^4}\,dx$

Consider $y=\dfrac{1}{(3-2x)^3}$

$\dfrac{dy}{dx}=\dfrac{6}{(3-2x)^4}$

So $\displaystyle\int_{\frac{1}{2}}^{1}\frac{12}{(3-2x)^4}\,dx=\left[\frac{2}{(3-2x)^3}\right]_{\frac{1}{2}}^{1}$

$=2-\tfrac{1}{4}=\tfrac{7}{4}$

c $\displaystyle\int_{\frac{2\pi}{9}}^{\frac{5\pi}{18}}\sec^2(\pi-3x)\,dx=\left[-\tfrac{1}{3}\tan(\pi-3x)\right]_{\frac{2\pi}{9}}^{\frac{5\pi}{18}}$

$=\left(-\dfrac{1}{3}\tan\left(\pi-\dfrac{15\pi}{18}\right)\right)-\left(-\dfrac{1}{3}\tan\left(\pi-\dfrac{6\pi}{9}\right)\right)$

$=\left(-\dfrac{1}{3}\tan\dfrac{\pi}{6}\right)-\left(-\dfrac{1}{3}\tan\dfrac{\pi}{3}\right)$

$=\left(-\dfrac{1}{3}\times\dfrac{1}{\sqrt{3}}\right)-\left(-\dfrac{1}{3}\times\sqrt{3}\right)$

$=-\dfrac{\sqrt{3}}{9}+\dfrac{\sqrt{3}}{3}=\dfrac{2\sqrt{3}}{9}$

d $\displaystyle\int_{2}^{3}\frac{5}{7-2x}\,dx=\left[-\tfrac{5}{2}\ln|7-2x|\right]_{2}^{3}$

$=\left(-\tfrac{5}{2}\ln 1\right)-\left(-\tfrac{5}{2}\ln 3\right)$

$=\tfrac{5}{2}\ln 3$

6 $\displaystyle\int_{3}^{b}(2x-6)^2\,dx=\int_{3}^{b}(4x^2-24x+36)\,dx$

$\left[\dfrac{4x^3}{3}-12x^2+36x\right]_{3}^{b}=36$

$\left(\dfrac{4b^3}{3}-12b^2+36b\right)-(36-108+108)=36$

$\dfrac{4b^3}{3}-12b^2+36b-72=0$

$b^3-9b^2+27b-54=0$

$(b-6)(b^2-3b+9)=0$

$b=6$ since $b^2-3b+9>0$.

7 $\displaystyle\int_{e^2}^{e^8}\frac{1}{kx}\,dx=\left[\frac{1}{k}\ln x\right]_{e^2}^{e^8}=\frac{1}{4}$

$\dfrac{8}{k}-\dfrac{2}{k}=\dfrac{1}{4}$

$k=32-8=24$

Challenge

$\displaystyle\int_{5}^{11}\frac{1}{ax+b}\,dx=\left[\frac{1}{a}\ln|ax+b|+\frac{1}{a}\ln k\right]_{5}^{11}$

where $\dfrac{1}{a}\ln k$ is a constant

$=\dfrac{1}{a}\Big[\ln k\,|ax+b|\Big]_{5}^{11}$

$=\dfrac{1}{a}(\ln k\,|11a+b|-\ln k\,|5a+b|)$

$=\dfrac{1}{a}(\ln k\,|11a+b|-k\ln|5a+b|)$

So $\ln k\,|11a+b|-\ln k\,|5a+b|=\ln\left(\dfrac{41}{17}\right)$

$\ln\left|\dfrac{11a+b}{5a+b}\right|=\ln\left(\dfrac{41}{17}\right)$

$\dfrac{11a+b}{5a+b}=\pm\dfrac{41}{17}$

Case 1:

$\dfrac{11a+b}{5a+b}=\dfrac{41}{17}$

$187a+17b=205a+41b$

$18a=-24b$

$3a=-4b$

So a must be a multiple of 4 between 0 and 10.

$a=4\Rightarrow b=-3$

$a=8\Rightarrow b=-6$

Case 2:

$\dfrac{11a+b}{5a+b}=-\dfrac{41}{17}$

$187a+17b=-205a-41b$

$392a=-58b$

$b=-\dfrac{196}{29}a$

But this cannot be an integer, since $a<29$, so case 2 gives no possible solutions.

Therefore the only two possible solutions are $a=4,\ b=-3$ and $a=8,\ b=-6$.