Chapter review

1 a $y = \ln x^2 = 2\ln x$

(using properties of logs)

$\therefore \frac{dy}{dx} = 2 \times \frac{1}{x} = \frac{2}{x}$

Alternative method:

When $y = \ln f(x)$, $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$

(by the chain rule)

$\therefore y = \ln x^2 \Rightarrow \frac{dy}{dx} = \frac{2x}{x^2} = \frac{2}{x}$

b $y = x^2 \sin 3x$

Using the product rule,

$\frac{dy}{dx} = x^2(3\cos 3x) + (\sin 3x) \times 2x$

$= 3x^2 \cos 3x + 2x \sin 3x$

2 a $2y = x - \sin x \cos x$

$\therefore y = \frac{x}{2} - \frac{1}{2}\sin x \cos x$

Using the product rule,

$\frac{dy}{dx} = \frac{1}{2} - \frac{1}{2}\left(\sin x(-\sin x) + \cos x \cos x\right)$

$= \frac{1}{2} + \frac{1}{2}\sin^2 x - \frac{1}{2}\cos^2 x$

$= \frac{1}{2}(1 - \cos^2 x) + \frac{1}{2}\sin^2 x$

$= \frac{1}{2}\sin^2 x + \frac{1}{2}\sin^2 x$

$= \sin^2 x$

b $y = \frac{x}{2} - \frac{1}{2}\sin x \cos x$

$\frac{dy}{dx} = \sin^2 x$

$\frac{d^2y}{dx^2} = 2\sin x \cos x = \sin 2x$

At points of inflection $\frac{d^2y}{dx^2} = 0$

i.e. $\sin 2x = 0$

$2x = \pi, 2\pi$ or 3π

$x = \frac{\pi}{2}, \pi$ or $\frac{3\pi}{2}$

When $x = \frac{\pi}{2}$, $y = \frac{\pi}{4}$

At $x = \frac{\pi}{3}$, $\frac{d^2y}{dx^2} > 0$; at $x = \frac{3\pi}{4}$, $\frac{d^2y}{dx^2} < 0$

So $\frac{d^2y}{dx^2}$ changes sign either side of $x = \frac{\pi}{2}$

When $x = \pi$, $y = \frac{\pi}{2}$

At $x = \frac{3\pi}{4}$, $\frac{d^2y}{dx^2} < 0$; at $x = \frac{5\pi}{4}$, $\frac{d^2y}{dx^2} > 0$

So $\frac{d^2y}{dx^2}$ changes sign either side of $x = \pi$

When $x = \frac{3\pi}{2}$, $y = \frac{3\pi}{4}$

At $x = \frac{5\pi}{4}$, $\frac{d^2y}{dx^2} > 0$; at $x = \frac{7\pi}{4}$, $\frac{d^2y}{dx^2} < 0$

So $\frac{d^2y}{dx^2}$ changes sign either side of $x = \frac{3\pi}{2}$

Hence the points of inflection are

$\left(\frac{\pi}{2}, \frac{\pi}{4}\right)$, $\left(\pi, \frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2}, \frac{3\pi}{4}\right)$

3 a $y = \dfrac{\sin x}{x}$

Using the quotient rule:

$$\frac{dy}{dx} = \frac{x\cos x - \sin x \times 1}{x^2} = \frac{x\cos x - \sin x}{x^2}$$

b $y = \ln\dfrac{1}{x^2+9} = \ln 1 - \ln(x^2+9) = -\ln(x^2+9)$

(by the laws of logarithms)

Using the chain rule:

$$\frac{dy}{dx} = -\frac{1}{x^2+9} \times 2x = -\frac{2x}{x^2+9}$$

4 a $f(x) = \dfrac{x}{x^2+2}$

$$f'(x) = \frac{(x^2+2)\times 1 - x \times 2x}{(x^2+2)^2} = \frac{2-x^2}{(x^2+2)^2}$$

The function is increasing when $f'(x) \geqslant 0$

i.e. $\dfrac{2-x^2}{(x^2+2)^2} \geqslant 0$

$x^2 \leqslant 2$

$-\sqrt{2} \leqslant x \leqslant \sqrt{2}$

Hence $f(x)$ is increasing on the interval $[-k, k]$ where $k = \sqrt{2}$.

b
$$f''(x) = \frac{-2x(x^2+2)^2 - 4x(x^2+2)(2-x^2)}{(x^2+2)^4} = \frac{2x(x^2+2)\left(-(x^2+2)-2(2-x^2)\right)}{(x^2+2)^4} = \frac{2x(x^2+2)(x^2-6)}{(x^2+2)^4}$$

$f''(x)$ changes sign when the numerator $2x(x^2+2)(x^2-6)$ is zero

i.e. at $x = 0$ and $x = \pm\sqrt{6}$

where $y = 0$ and $y = \dfrac{\pm\sqrt{6}}{6+2}$

Points of inflection are

$(0, 0)$ and $\left(\pm\sqrt{6}, \pm\dfrac{\sqrt{6}}{8}\right)$

5 a $f(x) = 12\ln x + x^{\frac{3}{2}}, \quad x > 0$

$$f'(x) = \frac{12}{x} + \frac{3}{2}x^{\frac{1}{2}} = \frac{12}{x} + \frac{3}{2}\sqrt{x}$$

$f(x)$ is an increasing function when $f'(x) \geq 0$

As $x > 0$, $\dfrac{12}{x} + \dfrac{3}{2}\sqrt{x}$ is always positive.

$\therefore$ $f(x)$ is increasing for all $x > 0$.

b $f''(x) = -\dfrac{12}{x^2} + \dfrac{3}{4}x^{-\frac{1}{2}} = -\dfrac{12}{x^2} + \dfrac{3}{4\sqrt{x}}$

At a point of inflection $f''(x) = 0$

$-\dfrac{12}{x^2} + \dfrac{3}{4\sqrt{x}} = 0$

$\dfrac{12}{x^2} = \dfrac{3}{4\sqrt{x}}$

$x^2 = 16\sqrt{x}$

$x^{\frac{3}{2}} = 16$

$x = \sqrt[3]{256}$

$$f\left(\sqrt[3]{256}\right) = 12\ln(256)^{\frac{1}{3}} + 256^{\frac{1}{2}} = 4\ln 256 + 16 = 4\ln 2^8 + 16 = 32\ln 2 + 16$$

Coordinates of the point of inflection are $\left(\sqrt[3]{256}, 32\ln 2 + 16\right)$

6 $y = \cos^2 x + \sin x$

$\frac{dy}{dx} = -2\cos x \sin x + \cos x$

$= \cos x(1 - 2\sin x)$

At stationary points $\frac{dy}{dx} = 0$

$\cos x(1 - 2\sin x) = 0$

$\cos x = 0$ or $\sin x = \frac{1}{2}$

Solutions in the interval (0, 2π] are

$x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$ and $\frac{3\pi}{2}$

$x = \frac{\pi}{6} \Rightarrow y = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}$

$x = \frac{\pi}{2} \Rightarrow y = 1$

$x = \frac{5\pi}{6} \Rightarrow y = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}$

$x = \frac{3\pi}{2} \Rightarrow y = -1$

So the stationary points are

$\left(\frac{\pi}{6}, \frac{5}{4}\right), \left(\frac{\pi}{2}, 1\right), \left(\frac{5\pi}{6}, \frac{5}{4}\right)$ and $\left(\frac{3\pi}{2}, -1\right)$

7 $y = x\sqrt{\sin x} = x(\sin x)^{\frac{1}{2}}$

$\frac{dy}{dx} = x \times \frac{1}{2}(\sin x)^{-\frac{1}{2}} \cos x + (\sin x)^{\frac{1}{2}} \times 1$

$= \frac{1}{2}(\sin x)^{-\frac{1}{2}}(x\cos x + 2\sin x)$

At the maximum point $\frac{dy}{dx} = 0$

$\frac{1}{2}(\sin x)^{-\frac{1}{2}}(x\cos x + 2\sin x) = 0$

$\therefore\ x\cos x + 2\sin x = 0$

(as $(\sin x)^{-\frac{1}{2}} = \frac{1}{\sqrt{\sin x}} \neq 0$)

Dividing through by cos x gives

$x + 2\tan x = 0$

So the x-coordinate of the maximum point satisfies $2\tan x + x = 0$.

8 **a** $f(x) = e^{0.5x} - x^2$

$f'(x) = 0.5e^{0.5x} - 2x$

b $f'(6) = -1.957... < 0$

$f'(7) = 2.557... > 0$

As the sign changes between $x = 6$ and $x = 7$ and $f'(x)$ is continuous, $f'(x) = 0$ has a root p between 6 and 7.

Therefore $y = f(x)$ has a stationary point at $x = p$ where $6 < p < 7$.

9 **a** $f(x) = e^{2x}\sin 2x$

$f'(x) = e^{2x}(2\cos 2x) + \sin 2x(2e^{2x})$

$= 2e^{2x}(\cos 2x + \sin 2x)$

At turning points $f'(x) = 0$

$2e^{2x}(\cos 2x + \sin 2x) = 0$

$\cos 2x + \sin 2x = 0$

$\sin 2x = -\cos 2x$

Divide both sides by $\cos 2x$:

$\tan 2x = -1$

$2x = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$

$\therefore x = \frac{3\pi}{8}$ or $\frac{7\pi}{8}$ (in the interval $0 < x < \pi$)

When $x = \frac{3\pi}{8}$, $y = \frac{1}{\sqrt{2}}e^{\frac{3\pi}{4}}$

When $x = \frac{7\pi}{8}$, $y = -\frac{1}{\sqrt{2}}e^{\frac{7\pi}{4}}$

So the coordinates of the turning points are $\left(\frac{3\pi}{8}, \frac{1}{\sqrt{2}}e^{\frac{3\pi}{4}}\right)$ and $\left(\frac{7\pi}{8}, -\frac{1}{\sqrt{2}}e^{\frac{7\pi}{4}}\right)$.

9 b $f'(x) = 2e^{2x}(\cos 2x + \sin 2x)$

$$f''(x) = 2e^{2x}(-2\sin 2x + 2\cos 2x) + 4e^{2x}(\cos 2x + \sin 2x)$$
$$= e^{2x}(-4\sin 2x + 4\cos 2x + 4\cos 2x + 4\sin 2x)$$
$$= 8e^{2x}\cos 2x$$

c $f''\left(\frac{3\pi}{8}\right) = 8e^{\frac{3\pi}{4}}\cos\frac{3\pi}{4}$

$$= 8e^{\frac{3\pi}{4}}\left(-\frac{\sqrt{2}}{2}\right) = -4\sqrt{2}\,e^{\frac{3\pi}{4}} < 0$$

$\therefore \left(\frac{3\pi}{8}, \frac{1}{\sqrt{2}}e^{\frac{3\pi}{4}}\right)$ is a maximum.

$f''\left(\frac{7\pi}{8}\right) = 8e^{\frac{7\pi}{4}}\cos\frac{7\pi}{4}$

$$= 8e^{\frac{7\pi}{4}}\left(\frac{\sqrt{2}}{2}\right) = 4\sqrt{2}\,e^{\frac{7\pi}{4}} > 0$$

$\therefore \left(\frac{7\pi}{8}, -\frac{1}{\sqrt{2}}e^{\frac{7\pi}{4}}\right)$ is a minimum.

d At points of inflection $f''(x) = 0$

$8e^{2x}\cos 2x = 0$

$\cos 2x = 0$

$2x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

$\therefore x = \frac{\pi}{4}$ or $\frac{3\pi}{4}$

When $x = \frac{\pi}{4}$, $y = e^{\frac{\pi}{2}}\sin\frac{\pi}{2} = e^{\frac{\pi}{2}}$

When $x = \frac{3\pi}{4}$, $y = e^{\frac{3\pi}{2}}\sin\frac{3\pi}{2} = -e^{\frac{3\pi}{2}}$

Points of inflection are

$\left(\frac{\pi}{4}, e^{\frac{\pi}{2}}\right)$ and $\left(\frac{3\pi}{4}, -e^{\frac{3\pi}{2}}\right)$.

10 $y = 2e^x + 3x^2 + 2$

$\frac{dy}{dx} = 2e^x + 6x$

When $x = 0$, $y = 4$ and $\frac{dy}{dx} = 2$

Equation of normal at (0, 4) is

$y - 4 = -\frac{1}{2}(x - 0)$

$2y - 8 = -x$

or $x + 2y - 8 = 0$

11 a $f(x) = 3\ln x + \frac{1}{x}$

$f'(x) = \frac{3}{x} - \frac{1}{x^2}$

At a stationary point $\frac{dy}{dx} = 0$

$\frac{3}{x} - \frac{1}{x^2} = 0$

$3x - 1 = 0$

$x = \frac{1}{3}$

So the x-coordinate of the stationary point P is $\frac{1}{3}$

b At the point Q, $x = 1$ so $y = f(1) = 1$

The gradient of the curve at point Q is $f'(1) = 3 - 1 = 2$

So the gradient of the normal to the curve at Q is $-\frac{1}{2}$

Equation of the normal to C at Q is

$y - 1 = -\frac{1}{2}(x - 1)$

i.e. $y = -\frac{1}{2}x + \frac{3}{2}$

12 a Let $f(x) = e^{2x}\cos x$

Then $f'(x) = e^{2x}(-\sin x) + \cos x(2e^{2x})$
$= e^{2x}(2\cos x - \sin x)$

Turning points occur when $f'(x) = 0$

$e^{2x}(2\cos x - \sin x) = 0$
$\sin x = 2\cos x$

Dividing both sides by $\cos x$ gives

$\tan x = 2$

b When $x = 0, y = f(0) = e^0 \cos 0 = 1$

The gradient of the curve at (0, 1) is
$f'(0) = e^0(2 - 0) = 2$

This is also the gradient of the tangent at (0, 1).

So the equation of the tangent at (0, 1) is

$y - 1 = 2(x - 0)$

$y = 2x + 1$

13 a $x = y^2 \ln y$

Using the product rule:

$$\frac{dx}{dy} = y^2\left(\frac{1}{y}\right) + \ln y \times 2y = y + 2y\ln y$$

b $$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{y + 2y\ln y}$$

When $y = e$,

$$\frac{dy}{dx} = \frac{1}{e + 2e\ln e} = \frac{1}{3e}$$

14 a $f(x) = (x^3 - 2x)e^{-x}$
$f'(x) = (x^3 - 2x)(-e^{-x}) + (3x^2 - 2)e^{-x}$
$= e^{-x}(-x^3 + 3x^2 + 2x - 2)$

b When $x = 0$, $f'(x) = -2$
Gradient of normal is $\frac{1}{2}$
∴ equation of normal to the curve at the origin is
$y = \frac{1}{2}x$
This line will intersect the curve again when
$\frac{1}{2}x = (x^3 - 2x)e^{-x}$
$1 = 2(x^2 - 2)e^{-x}$
$e^x = 2x^2 - 4$
$2x^2 = e^x + 4$

Challenge

a $f(x) = x(1 + x)\ln x = (x + x^2)\ln x$
$$f'(x) = (x + x^2) \times \frac{1}{x} + \ln x \times (1 + 2x)$$
$= 1 + x + (1 + 2x)\ln x$

b At minimum point A, $f'(x) = 0$
$1 + x + (1 + 2x)\ln x = 0$
$(1 + 2x)\ln x = -(1 + x)$
$$\ln x = -\frac{1 + x}{1 + 2x}$$
So x-coordinate of A is the solution to the equation $x = e^{-\frac{1+x}{1+2x}}$