Exercise 6F

1 a $y = \tan 3x$

Using the result

$y = \tan kx \Rightarrow \frac{dy}{dx} = k\sec^2 kx$

$\frac{dy}{dx} = 3\sec^2 3x$

b $y = 4\tan^3 x$

Let $u = \tan x$; then $y = 4u^3$

$\frac{du}{dx} = \sec^2 x$ and $\frac{dy}{du} = 12u^2$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 12u^2 \sec^2 x$$
$$= 12\tan^2 x \sec^2 x$$

c $y = \tan(x-1)$

$\frac{dy}{dx} = \sec^2(x-1)$

d $y = x^2 \tan \frac{1}{2}x + \tan\left(x - \frac{1}{2}\right)$

The first term is a product with $u = x^2$ and $v = \tan\frac{1}{2}x$

$\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = \frac{1}{2}\sec^2 \frac{1}{2}x$

Using the product rule for the first term:

$$\frac{dy}{dx} = x^2\left(\tfrac{1}{2}\sec^2 \tfrac{1}{2}x\right) + \tan\tfrac{1}{2}x \times 2x + \sec^2\left(x - \tfrac{1}{2}\right)$$
$$= \tfrac{1}{2}x^2 \sec^2 \tfrac{1}{2}x + 2x\tan\tfrac{1}{2}x + \sec^2\left(x - \tfrac{1}{2}\right)$$

2 a $y = \cot 4x$

Let $u = 4x$; then $y = \cot u$

$\frac{du}{dx} = 4$ and $\frac{dy}{du} = -\text{cosec}^2 u$

Using the chain rule,

$\frac{dy}{dx} = -\text{cosec}^2 u \times 4 = -4\,\text{cosec}^2 4x$

b $y = \sec 5x$

Let $u = 5x$; then $y = \sec u$

$\frac{du}{dx} = 5$ and $\frac{dy}{du} = \sec u \tan u$

Using the chain rule,

$\frac{dy}{dx} = 5\sec u \tan u = 5\sec 5x \tan 5x$

2 c $y = \operatorname{cosec} 4x$

Let $u = 4x$; then $y = \operatorname{cosec} u$

$\frac{du}{dx} = 4$ and $\frac{dy}{du} = -\operatorname{cosec} u \cot u$

Using the chain rule,

$$\frac{dy}{dx} = -4\operatorname{cosec} u \cot u$$
$$= -4\operatorname{cosec} 4x \cot 4x$$

d $y = \sec^2 3x = (\sec 3x)^2$

Let $u = \sec 3x$; then $y = u^2$

$\frac{du}{dx} = 3\sec 3x \tan 3x$ and $\frac{dy}{du} = 2u$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= 2u \times 3\sec 3x \tan 3x$$
$$= 2\sec 3x \times 3\sec 3x \tan 3x$$
$$= 6\sec^2 3x \tan 3x$$

e $y = x\cot 3x$

This is a product, so let
$u = x$ and $v = \cot 3x$
and use the product rule.

$\frac{du}{dx} = 1$ and $\frac{dv}{dx} = -3\operatorname{cosec}^2 3x$

$$\frac{dy}{dx} = x(-3\operatorname{cosec}^2 3x) + \cot 3x \times 1$$
$$= \cot 3x - 3x\operatorname{cosec}^2 3x$$

f $y = \frac{\sec^2 x}{x}$

This is a quotient, so let
$u = \sec^2 x$ and $v = x$
and use the quotient rule.

$\frac{du}{dx} = 2\sec x(\sec x \tan x)$ and $\frac{dv}{dx} = 1$

$$\frac{dy}{dx} = \frac{x(2\sec^2 x \tan x) - \sec^2 x \times 1}{x^2}$$
$$= \frac{\sec^2 x(2x\tan x - 1)}{x^2}$$

g $y = \operatorname{cosec}^3 2x$

Let $u = \operatorname{cosec} 2x$; then $y = u^3$

$\frac{du}{dx} = -2\operatorname{cosec} 2x \cot 2x$ and $\frac{dy}{du} = 3u^2$

Using the chain rule,

$$\frac{dy}{dx} = 3u^2(-2\operatorname{cosec} 2x \cot 2x)$$
$$= -6\operatorname{cosec}^2 2x \operatorname{cosec} 2x \cot 2x$$
$$= -6\operatorname{cosec}^3 2x \cot 2x$$

h $y = \cot^2(2x-1)$

Let $u = \cot(2x-1)$; then $y = u^2$

$\frac{du}{dx} = -2\operatorname{cosec}^2(2x-1)$ and $\frac{dy}{du} = 2u$

Using the chain rule,

$$\frac{dy}{dx} = 2u\left(-2\operatorname{cosec}^2(2x-1)\right)$$
$$= -4\cot(2x-1)\operatorname{cosec}^2(2x-1)$$

3 a $f(x) = (\sec x)^{\frac{1}{2}}$

Using the chain rule,

$$f'(x) = \tfrac{1}{2}(\sec x)^{-\frac{1}{2}} \times \sec x \tan x$$
$$= \tfrac{1}{2}(\sec x)^{\frac{1}{2}} \tan x$$

3 b $f(x) = \sqrt{\cot x} = (\cot x)^{\frac{1}{2}}$

Using the chain rule,

$f'(x) = \frac{1}{2}(\cot x)^{-\frac{1}{2}} \times (-\operatorname{cosec}^2 x)$

$= -\frac{1}{2}(\cot x)^{-\frac{1}{2}} \operatorname{cosec}^2 x$

c $f(x) = \operatorname{cosec}^2 x = (\operatorname{cosec} x)^2$

Using the chain rule,

$f'(x) = 2(\operatorname{cosec} x)^1(-\operatorname{cosec} x \cot x)$

$= -2\operatorname{cosec}^2 x \cot x$

d $f(x) = \tan^2 x = (\tan x)^2$

Using the chain rule,

$f'(x) = 2\tan x \times \sec^2 x = 2\tan x \sec^2 x$

e $f(x) = \sec^3 x = (\sec x)^3$

Using the chain rule,

$f'(x) = 3(\sec x)^2 \sec x \tan x = 3\sec^3 x \tan x$

f $f(x) = \cot^3 x = (\cot x)^3$

Using the chain rule,

$f'(x) = 3(\cot x)^2(-\operatorname{cosec}^2 x)$

$= -3\cot^2 x \operatorname{cosec}^2 x$

4 a $f(x) = x^2 \sec 3x$

Let $u = x^2$ and $v = \sec 3x$

$\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = 3\sec 3x \tan 3x$

Using the product rule,

$f'(x) = 3x^2 \sec 3x \tan 3x + 2x \sec 3x$

b $f(x) = \frac{\tan 2x}{x}$

Let $u = \tan 2x$ and $v = x$

$\frac{du}{dx} = 2\sec^2 2x$ and $\frac{dv}{dx} = 1$

Using the quotient rule,

$f'(x) = \frac{2x\sec^2 2x - \tan 2x}{x^2}$

c $f(x) = \frac{x^2}{\tan x}$

Let $u = x^2$ and $v = \tan x$

$\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = \sec^2 x$

Using the quotient rule,

$f'(x) = \frac{2x\tan x - x^2 \sec^2 x}{\tan^2 x}$

d $f(x) = e^x \sec 3x$

Let $u = e^x$ and $v = \sec 3x$

$\frac{du}{dx} = e^x$ and $\frac{dv}{dx} = 3\sec 3x \tan 3x$

Using the product rule,

$f'(x) = 3e^x \sec 3x \tan 3x + e^x \sec 3x$

$= e^x \sec 3x(3\tan 3x + 1)$

e $f(x) = \frac{\ln x}{\tan x}$

Let $u = \ln x$ and $v = \tan x$

$\frac{du}{dx} = \frac{1}{x}$ and $\frac{dv}{dx} = \sec^2 x$

Using the quotient rule,

$f'(x) = \frac{\left(\frac{1}{x}\right)\tan x - \ln x \sec^2 x}{\tan^2 x}$

$= \frac{\tan x - x\ln x \sec^2 x}{x\tan^2 x}$

f $f(x) = \frac{e^{\tan x}}{\cos x}$

Let $u = e^{\tan x}$ and $v = \cos x$

$\frac{du}{dx} = e^{\tan x}\sec^2 x$ and $\frac{dv}{dx} = -\sin x$

Using the quotient rule,

$f'(x) = \frac{e^{\tan x}\sec^2 x \cos x - e^{\tan x}(-\sin x)}{\cos^2 x}$

$= \frac{e^{\tan x}\sec x + e^{\tan x}\sin x}{\cos^2 x}$

$= \frac{e^{\tan x}(\sec x + \sin x)}{\cos^2 x}$

$= e^{\tan x}(\sec^3 x + \sec x \tan x)$

$= e^{\tan x}\sec x(\sec^2 x + \tan x)$

5 a $y = \dfrac{1}{\cos x \sin x} = \sec x \operatorname{cosec} x$

Let $u = \sec x$ and $v = \operatorname{cosec} x$

$\dfrac{du}{dx} = \sec x \tan x$ and $\dfrac{dv}{dx} = -\operatorname{cosec} x \cot x$

Using the product rule,

$$\frac{dy}{dx} = \sec x(-\operatorname{cosec} x \cot x) + \operatorname{cosec} x(\sec x \tan x)$$
$$= -\frac{\cos x}{\cos x \sin x \sin x} + \frac{\sin x}{\sin x \cos x \cos x}$$
$$= -\frac{1}{\sin^2 x} + \frac{1}{\cos^2 x}$$

Alternative solution:

$y = \dfrac{1}{\cos x \sin x} = \dfrac{2}{\sin 2x} = 2\operatorname{cosec} 2x$

(because $\sin 2x = 2\sin x \cos x$)

$\dfrac{dy}{dx} = -4\operatorname{cosec} 2x \cot 2x$

b At stationary points $\dfrac{dy}{dx} = 0$

$$\frac{1}{\cos^2 x} - \frac{1}{\sin^2 x} = 0$$
$$\frac{1}{\cos^2 x} = \frac{1}{\sin^2 x}$$
$$\tan^2 x = 1$$
$$\tan x = \pm 1$$

In the interval $0 < x \leqslant \pi$

there are two solutions, $\dfrac{\pi}{4}$ and $\dfrac{3\pi}{4}$.

So the number of stationary points is 2.

Alternative solution:

$-4\operatorname{cosec} 2x \cot 2x = 0$

$\operatorname{cosec} 2x \neq 0$

but $\cot 2x = 0$ has two solutions in the interval $0 < x \leqslant \pi$.

So there are 2 stationary points.

c When $x = \dfrac{\pi}{3}$,

$$y = \frac{1}{\cos\frac{\pi}{3}\sin\frac{\pi}{3}} = \frac{1}{\frac{1}{2}\times\frac{\sqrt{3}}{2}} = \frac{4}{\sqrt{3}} = \frac{4\sqrt{3}}{3}$$

$$\frac{dy}{dx} = -\frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2} + \frac{1}{\left(\frac{1}{2}\right)^2} = -\frac{4}{3} + 4 = \frac{8}{3}$$

or, using the alternative expression,

$$\frac{dy}{dx} = -4 \times \frac{2}{\sqrt{3}} \times \left(-\frac{1}{\sqrt{3}}\right) = \frac{8}{3}$$

Equation of tangent is

$$y - \frac{4\sqrt{3}}{3} = \frac{8}{3}\left(x - \frac{\pi}{3}\right)$$
$$3y - 4\sqrt{3} = 8x - \frac{8\pi}{3}$$
$$24x - 9y + 12\sqrt{3} - 8\pi = 0$$

This is in the required form $ax + by + c = 0$

With $a = 24$, $b = -9$ and $c = 12\sqrt{3} - 8\pi$.

6 $y = \sec x = \dfrac{1}{\cos x}$

Let $u = 1$ and $v = \cos x$

$\dfrac{du}{dx} = 0$ and $\dfrac{dv}{dx} = -\sin x$

Using the quotient rule,

$$\frac{dy}{dx} = \frac{\cos x \times 0 - 1 \times (-\sin x)}{\cos^2 x}$$
$$= \frac{\sin x}{\cos^2 x} = \sec x \tan x$$

7 $y = \cot x = \dfrac{1}{\tan x}$

Let $u = 1$ and $v = \tan x$

$\dfrac{du}{dx} = 0$ and $\dfrac{dv}{dx} = \sec^2 x$

Using the quotient rule,

$$\frac{dy}{dx} = \frac{\tan x \times 0 - 1 \times \sec^2 x}{\tan^2 x} = -\frac{\sec^2 x}{\tan^2 x}$$
$$= -\frac{\frac{1}{\cos^2 x}}{\frac{\sin^2 x}{\cos^2 x}} = -\frac{1}{\sin^2 x} = -\operatorname{cosec}^2 x$$

8 a Let $y = \arccos x$

So $x = \cos y$

$\frac{dx}{dy} = -\sin y$

$\frac{dy}{dx} = -\frac{1}{\sin y}$

$\sin^2 y + \cos^2 y = 1$

$\sin y = \sqrt{1-\cos^2 y} = \sqrt{1-x^2}$

$\therefore \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$

b Let $y = \arctan x$

So $x = \tan y$

$\frac{dx}{dy} = \sec^2 y$

$\frac{dy}{dx} = \frac{1}{\sec^2 y}$

$= \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$

9 a Let $y = \arccos 2x$

Let $t = 2x$; then $y = \arccos t$

$\frac{dt}{dx} = 2$ and $\frac{dy}{dt} = \frac{-1}{\sqrt{1-t^2}}$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= \frac{-1}{\sqrt{1-t^2}} \times 2$

$= \frac{-2}{\sqrt{1-4x^2}}$

b Let $y = \arctan \frac{x}{2}$

Let $t = \frac{x}{2}$; then $y = \arctan t$

$\frac{dt}{dx} = \frac{1}{2}$ and $\frac{dy}{dt} = \frac{1}{1+t^2}$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= \frac{1}{1+t^2} \times \frac{1}{2} = \frac{1}{2\left(1+\frac{x^2}{4}\right)} = \frac{1}{2+\frac{x^2}{2}}$

$= \frac{2}{4+x^2}$

c Let $y = \arcsin 3x$

So $\sin y = 3x$

$x = \frac{\sin y}{3}$

$\frac{dx}{dy} = \frac{\cos y}{3}$

$\frac{dy}{dx} = \frac{3}{\cos y}$

$\sin^2 y + \cos^2 y = 1$

$\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-(3x)^2} = \sqrt{1-9x^2}$

$\therefore \frac{dy}{dx} = \frac{3}{\sqrt{1-9x^2}}$

d Let $y = \text{arccot}\, x$

So $x = \cot y$

$\frac{dx}{dy} = -\text{cosec}^2 y$

$\frac{dy}{dx} = \frac{-1}{\text{cosec}^2 y}$

$= \frac{-1}{1+\cot^2 y} = \frac{-1}{1+x^2}$

9 e Let $y = \text{arcsec}\, x$

So $x = \sec y$

$$\frac{dx}{dy} = \sec y \tan y$$

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y}$$

$$1 + \tan^2 y = \sec^2 y$$

$$\tan y = \sqrt{\sec^2 y - 1} = \sqrt{x^2 - 1}$$

$$\therefore \frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}$$

f Let $y = \text{arccosec}\, x$

So $x = \text{cosec}\, y$

$$\frac{dx}{dy} = -\text{cosec}\, y \cot y$$

$$\frac{dy}{dx} = -\frac{1}{\text{cosec}\, y \cot y}$$

$$1 + \cot^2 y = \text{cosec}^2 y$$

$$\cot y = \sqrt{\text{cosec}^2 y - 1} = \sqrt{x^2 - 1}$$

$$\therefore \frac{dy}{dx} = \frac{-1}{x\sqrt{x^2-1}}$$

g Let $y = \arcsin\left(\frac{x}{x-1}\right)$

Let $u = \frac{x}{x-1}$

then $y = \arcsin u$, $\frac{dy}{du} = \frac{1}{\sqrt{1-u^2}}$

Using the quotient rule,

$$\frac{du}{dx} = \frac{(x-1)\times 1 - x\times 1}{(x-1)^2} = \frac{-1}{(x-1)^2}$$

$$\frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx} = \frac{1}{\sqrt{1-u^2}}\times\frac{-1}{(x-1)^2}$$

$$= \frac{1}{\sqrt{1-\frac{x^2}{(x-1)^2}}}\times\frac{-1}{(x-1)^2}$$

$$= \frac{1}{\sqrt{\frac{(x-1)^2-x^2}{(x-1)^2}}}\times\frac{-1}{(x-1)^2}$$

$$= \frac{1}{\frac{\sqrt{1-2x}}{x-1}}\times\frac{-1}{(x-1)^2} = \frac{-1}{(x-1)\sqrt{1-2x}}$$

h Let $y = \arccos x^2$

Let $t = x^2$ and $y = \arccos t$

$$\frac{dt}{dx} = 2x \text{ and } \frac{dy}{dt} = \frac{-1}{\sqrt{1-t^2}}$$

$$\frac{dy}{dx} = \frac{dy}{dt}\times\frac{dt}{dx}$$

$$= \frac{-1}{\sqrt{1-t^2}}\times 2x = \frac{-2x}{\sqrt{1-x^4}}$$

i Let $y = e^x \arccos x$

Using the product rule,

$$\frac{dy}{dx} = e^x\left(\frac{-1}{\sqrt{1-x^2}}\right) + e^x \arccos x$$

$$= e^x\left(\arccos x - \frac{1}{\sqrt{1-x^2}}\right)$$

j Let $y = \arcsin x \cos x$

Using the product rule,

$$\frac{dy}{dx} = \arcsin x\,(-\sin x) + \frac{1}{\sqrt{1-x^2}}\cos x$$

$$= \frac{\cos x}{\sqrt{1-x^2}} - \sin x \arcsin x$$

k Let $y = x^2 \arccos x$

Using the product rule,

$$\frac{dy}{dx} = x^2 \times \frac{-1}{\sqrt{1-x^2}} + 2x\arccos x$$

$$= 2x\arccos x - \frac{x^2}{\sqrt{1-x^2}}$$

$$= x\left(2\arccos x - \frac{x}{\sqrt{1-x^2}}\right)$$

9 l Let $y = e^{\arctan x}$

Let $u = \arctan x$; then $y = e^u$

$$\frac{dy}{du} = e^u \text{ and } \frac{du}{dx} = \frac{1}{1+x^2}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{e^u}{1+x^2} = \frac{e^{\arctan x}}{1+x^2}$$

10 a $y = \dfrac{\arctan 2x}{x}$

Let $u = \arctan 2x$ and $v = x$

$$\frac{du}{dx} = \frac{2}{1+4x^2} \text{ and } \frac{dv}{dx} = 1$$

Using the quotient rule,

$$\frac{dy}{dx} = \frac{x\left(\frac{2}{1+4x^2}\right) - 1 \times \arctan 2x}{x^2}$$

$$= \frac{2}{x(1+4x^2)} - \frac{\arctan 2x}{x^2}$$

When $x = \dfrac{\sqrt{3}}{2}$,

$$\frac{dy}{dx} = \frac{2}{\frac{\sqrt{3}}{2}\left(1+4\left(\frac{\sqrt{3}}{2}\right)^2\right)} - \frac{\arctan\sqrt{3}}{\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{2}{\frac{\sqrt{3}}{2} \times 4} - \frac{\frac{\pi}{3}}{\frac{3}{4}}$$

$$= \frac{1}{\sqrt{3}} - \frac{4\pi}{9} = \frac{3\sqrt{3} - 4\pi}{9}$$

b When $x = \dfrac{\sqrt{3}}{2}$,

$$y = \frac{\arctan\sqrt{3}}{\frac{\sqrt{3}}{2}} = \frac{\frac{\pi}{3}}{\frac{\sqrt{3}}{2}} = \frac{2\pi}{3\sqrt{3}} = \frac{2\sqrt{3}\pi}{9}$$

Equation of normal is

$$y - \frac{2\sqrt{3}\,\pi}{9} = \left(\frac{-9}{3\sqrt{3} - 4\pi}\right)\left(x - \frac{\sqrt{3}}{2}\right)$$

$$y = -\left(\frac{9}{3\sqrt{3} - 4\pi}\right)x + \frac{9\sqrt{3}}{6\sqrt{3} - 8\pi} + \frac{2\sqrt{3}\,\pi}{9}$$

11 $x = (\arccos y)^2$

$\sqrt{x} = \arccos y$

$y = \cos(\sqrt{x})$

Let $u = \sqrt{x} = x^{\frac{1}{2}}$; then $y = \cos u$

$$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} \text{ and } \frac{dy}{du} = -\sin u$$

Using the chain rule,

$$\frac{dy}{dx} = -\sin u \times \frac{1}{2\sqrt{x}} = -\frac{\sin\sqrt{x}}{2\sqrt{x}}$$

$$\sin^2\sqrt{x} + \cos^2\sqrt{x} = 1$$

$$\sin\sqrt{x} = \sqrt{1 - \cos^2\sqrt{x}}$$

$$\therefore \frac{dy}{dx} = -\frac{\sqrt{1 - \cos^2\sqrt{x}}}{2\sqrt{x}}$$

12 a $x = \operatorname{cosec} 5y$

$$\frac{dx}{dy} = -5\operatorname{cosec} 5y \cot 5y$$

$$\frac{dy}{dx} = -\frac{1}{5\operatorname{cosec} 5y \cot 5y}$$

b $1 + \cot^2 5y = \operatorname{cosec}^2 5y$

$$\cot 5y = \sqrt{\operatorname{cosec}^2 5y - 1} = \sqrt{x^2 - 1}$$

$$\therefore \frac{dy}{dx} = -\frac{1}{5x\sqrt{x^2 - 1}}$$