Exercise 6E

1 a Let $y = \frac{5x}{x+1}$

Let $u = 5x$ and $v = x + 1$

Then $\frac{du}{dx} = 5$ and $\frac{dv}{dx} = 1$

Using $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{(x+1)\times 5 - 5x\times 1}{(x+1)^2} = \frac{5}{(x+1)^2}$$

b Let $y = \frac{2x}{3x-2}$

Let $u = 2x$ and $v = 3x - 2$

Then $\frac{du}{dx} = 2$ and $\frac{dv}{dx} = 3$

Using $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{(3x-2)\times 2 - 2x\times 3}{(3x-2)^2}$$
$$= \frac{6x-4-6x}{(3x-2)^2} = -\frac{4}{(3x-2)^2}$$

c Let $y = \frac{x+3}{2x+1}$

Let $u = x + 3$ and $v = 2x + 1$

Then $\frac{du}{dx} = 1$ and $\frac{dv}{dx} = 2$

Using $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{(2x+1)\times 1 - (x+3)\times 2}{(2x+1)^2}$$
$$= \frac{2x+1-2x-6}{(2x+1)^2} = -\frac{5}{(2x+1)^2}$$

d Let $y = \frac{3x^2}{(2x-1)^2}$

Let $u = 3x^2$ and $v = (2x-1)^2$

Then $\frac{du}{dx} = 6x$ and $\frac{dv}{dx} = 4(2x-1)$

Using $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{(2x-1)^2\times 6x - 3x^2\times 4(2x-1)}{(2x-1)^4}$$
$$= \frac{6x(2x-1)\left((2x-1)-2x\right)}{(2x-1)^4}$$
$$= \frac{-6x(2x-1)}{(2x-1)^4} = -\frac{6x}{(2x-1)^3}$$

e Let $y = \frac{6x}{(5x+3)^{\frac{1}{2}}}$

Let $u = 6x$ and $v = (5x+3)^{\frac{1}{2}}$

Then $\frac{du}{dx} = 6$ and $\frac{dv}{dx} = \frac{5}{2}(5x+3)^{-\frac{1}{2}}$

Using $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{(5x+3)^{\frac{1}{2}}\times 6 - 6x\times \frac{5}{2}(5x+3)^{-\frac{1}{2}}}{\left((5x+3)^{\frac{1}{2}}\right)^2}$$
$$= \frac{3(5x+3)^{-\frac{1}{2}}\left(2(5x+3)-5x\right)}{(5x+3)}$$
$$= \frac{3(5x+3)^{-\frac{1}{2}}(10x+6-5x)}{(5x+3)} = \frac{3(5x+6)}{(5x+3)^{\frac{3}{2}}}$$

2 a Let $y = \dfrac{e^{4x}}{\cos x}$

Let $u = e^{4x}$ and $v = \cos x$

$\dfrac{du}{dx} = 4e^{4x}$ and $\dfrac{dv}{dx} = -\sin x$

Using $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{4e^{4x}\cos x - e^{4x}(-\sin x)}{\cos^2 x}$$

$$= \frac{e^{4x}(4\cos x + \sin x)}{\cos^2 x}$$

b Let $y = \dfrac{\ln x}{x+1}$

Let $u = \ln x$ and $v = x + 1$

$\dfrac{du}{dx} = \dfrac{1}{x}$ and $\dfrac{dv}{dx} = 1$

Using $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{\dfrac{(x+1)}{x} - \ln x}{(x+1)^2} = \frac{1}{x(x+1)} - \frac{\ln x}{(x+1)^2}$$

c Let $y = \dfrac{e^{-2x} + e^{2x}}{\ln x}$

Let $u = e^{-2x} + e^{2x}$ and $v = \ln x$

$\dfrac{du}{dx} = -2e^{-2x} + 2e^{2x}$ and $\dfrac{dv}{dx} = \dfrac{1}{x}$

Using $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{\ln x\,(2e^{2x} - e^{-2x}) - \dfrac{e^{-2x} + e^{2x}}{x}}{(\ln x)^2}$$

$$= \frac{2x\ln x\,(e^{2x} - e^{-2x}) - (e^{-2x} + e^{2x})}{x(\ln x)^2}$$

$$= \frac{e^{-2x}\left(2x(e^{4x} - 1)\ln x - e^{4x} - 1\right)}{x(\ln x)^2}$$

d Let $y = \dfrac{(e^x + 3)^3}{\cos x}$

Let $u = (e^x + 3)^3$ and $v = \cos x$

$\dfrac{du}{dx} = 3e^x(e^x + 3)^2$ and $\dfrac{dv}{dx} = -\sin x$

Using $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{3e^x(e^x + 3)^2\cos x - (-\sin x)(e^x + 3)^3}{\cos^2 x}$$

$$= \frac{(e^x + 3)^2\left(3e^x\cos x + (e^x + 3)\sin x\right)}{\cos^2 x}$$

e Let $y = \dfrac{\sin^2 x}{\ln x}$

Let $u = \sin^2 x$ and $v = \ln x$

$\dfrac{du}{dx} = 2\sin x\cos x$ and $\dfrac{dv}{dx} = \dfrac{1}{x}$

Using $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

$$\frac{dy}{dx} = \frac{\ln x(2\sin x\cos x) - \dfrac{1}{x}\sin^2 x}{(\ln x)^2}$$

$$= \frac{2\sin x\cos x}{\ln x} - \frac{\sin^2 x}{x(\ln x)^2}$$

3 $y=\dfrac{x}{3x+1}$

Let $u=x$ and $v=3x+1$

$\dfrac{du}{dx}=1$ and $\dfrac{dv}{dx}=3$

Using $\dfrac{dy}{dx}=\dfrac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}$

$\dfrac{dy}{dx}=\dfrac{(3x+1)-3x}{(3x+1)^2}=\dfrac{1}{(3x+1)^2}$

At the point $\left(1,\frac{1}{4}\right)$, $x=1$

so $\dfrac{dy}{dx}=\dfrac{1}{4^2}=\dfrac{1}{16}$

4 $y=\dfrac{x+3}{(2x+1)^{\frac{1}{2}}}$

Let $u=x+3$ and $v=(2x+1)^{\frac{1}{2}}$

$\dfrac{du}{dx}=1$ and $\dfrac{dv}{dx}=(2x+1)^{-\frac{1}{2}}$

Using $\dfrac{dy}{dx}=\dfrac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}$

$\dfrac{dy}{dx}=\dfrac{(2x+1)^{\frac{1}{2}}-(x+3)(2x+1)^{-\frac{1}{2}}}{2x+1}$

At the point (12, 3), $x=12$

so $\dfrac{dy}{dx}=\dfrac{25^{\frac{1}{2}}-(15\times 25^{-\frac{1}{2}})}{25}$

$=\dfrac{5-15\times\frac{1}{5}}{25}=\dfrac{2}{25}$

5 $y=\dfrac{e^{2x+3}}{x}$

Let $u=e^{2x+3}$ and $v=x$

$\dfrac{du}{dx}=2e^{2x+3}$ and $\dfrac{dv}{dx}=1$

Using $\dfrac{dy}{dx}=\dfrac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}$

$\dfrac{dy}{dx}=\dfrac{2xe^{2x+3}-e^{2x+3}}{x^2}=\dfrac{e^{2x+3}(2x-1)}{x^2}$

At stationary points $\dfrac{dy}{dx}=0$

so $2x-1=0$

$x=0.5$ and $y=2e^4$

There is one stationary point at $(0.5, 2e^4)$.

6 $y=\dfrac{e^{\frac{1}{3}x}}{x}$

Let $u=e^{\frac{1}{3}x}$ and $v=x$

$\dfrac{du}{dx}=\dfrac{1}{3}e^{\frac{1}{3}x}$ and $\dfrac{dv}{dx}=1$

Using $\dfrac{dy}{dx}=\dfrac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}$

$\dfrac{dy}{dx}=\dfrac{\frac{x}{3}e^{\frac{1}{3}x}-e^{\frac{1}{3}x}}{x^2}=\dfrac{e^{\frac{1}{3}x}\left(\frac{x}{3}-1\right)}{x^2}$

At the point $\left(3,\frac{1}{3}e\right)$, $x=3$ so $\dfrac{dy}{dx}=0$

Equation of tangent is

$y-\frac{1}{3}e=0(x-3)$

i.e. $y=\frac{1}{3}e$

7 $y=\dfrac{\ln x}{\sin 3x}$

Let $u=\ln x$ and $v=\sin 3x$

$\dfrac{du}{dx}=\dfrac{1}{x}$ and $\dfrac{dv}{dx}=3\cos 3x$

Using $\dfrac{dy}{dx}=\dfrac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}$

$\dfrac{dy}{dx}=\dfrac{\dfrac{\sin 3x}{x}-3\ln x\cos 3x}{\sin^2 3x}$

$=\dfrac{\sin 3x-3x\ln x\cos 3x}{x\sin^2 3x}$

When $x=\dfrac{\pi}{9}$,

$$\frac{dy}{dx}=\frac{\sin\frac{\pi}{3}-\frac{\pi}{3}\ln\left(\frac{\pi}{9}\right)\cos\frac{\pi}{3}}{\frac{\pi}{9}\sin^2\frac{\pi}{3}}$$

$$=\frac{\frac{\sqrt{3}}{2}-\frac{\pi}{6}\ln\left(\frac{\pi}{9}\right)}{\frac{3\pi}{36}}=\frac{18\sqrt{3}-6\pi\ln\left(\frac{\pi}{9}\right)}{3\pi}$$

8 a $x = \dfrac{e^y}{3+2y}$

When $y = 0$, $x = \dfrac{e^0}{3} = \dfrac{1}{3}$

Coordinates of P are $\left(\frac{1}{3}, 0\right)$.

b Let $u = e^y$ and $v = 3+2y$

$\dfrac{du}{dy} = e^y$ and $\dfrac{dv}{dy} = 2$

$\dfrac{dx}{dy} = \dfrac{e^y(3+2y) - 2e^y}{(3+2y)^2} = \dfrac{e^y(2y+1)}{(3+2y)^2}$

Gradient of normal to the curve is

$-\dfrac{1}{\frac{dy}{dx}} = -\dfrac{dx}{dy} = -\dfrac{e^y(2y+1)}{(3+2y)^2}$

Gradient of normal at $P\left(\frac{1}{3}, 0\right)$ is

$-\dfrac{e^0(2\times 0+1)}{3^2} = -\dfrac{1}{9}$

Equation of normal at P is

$y - 0 = -\frac{1}{9}\left(x - \frac{1}{3}\right)$

$y = -\frac{1}{9}x + \frac{1}{27}$

This is in the form $y = mx + c$ with $m = -\frac{1}{9}$ and $c = \frac{1}{27}$

9 Let $y = \dfrac{x^4}{\cos 3x}$

Let $u = x^4$ and $v = \cos 3x$

$\dfrac{du}{dx} = 4x^3$ and $\dfrac{dv}{dx} = -3\sin 3x$

Using $\dfrac{dy}{dx} = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$\dfrac{dy}{dx} = \dfrac{4x^3\cos 3x - x^4(-3\sin 3x)}{\cos^2 3x}$

$= \dfrac{x^3(4\cos 3x + 3x\sin 3x)}{\cos^2 3x}$

10 a $y = \dfrac{e^{2x}}{(x-2)^2}$

Let $u = e^{2x}$ and $v = (x-2)^2$

$\dfrac{du}{dx} = 2e^{2x}$ and $\dfrac{dv}{dx} = 2(x-2)$

Using $\dfrac{dy}{dx} = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$\dfrac{dy}{dx} = \dfrac{2(x-2)^2e^{2x} - 2e^{2x}(x-2)}{(x-2)^4}$

$= \dfrac{2e^{2x}(x-2)((x-2)-1)}{(x-2)^4}$

$= \dfrac{2e^{2x}(x-3)}{(x-2)^3}$

So $A = 2$, $B = 1$ and $C = 3$.

b When $x = 1$, $y = e^2$

and $\dfrac{dy}{dx} = \dfrac{2e^2(-2)}{-1} = 4e^2$

Equation of tangent is

$y - e^2 = 4e^2(x-1)$

$y = 4e^2x - 3e^2$

11 a $f(x) = \dfrac{2x}{x+5} + \dfrac{6x}{x^2+7x+10}$

$f(x) = \dfrac{2x}{x+5} + \dfrac{6x}{(x+2)(x+5)}$

$= \dfrac{2x(x+2)}{(x+2)(x+5)} + \dfrac{6x}{(x+2)(x+5)}$

$= \dfrac{2x^2+4x+6x}{(x+2)(x+5)} = \dfrac{2x^2+10x}{(x+2)(x+5)}$

$= \dfrac{2x(x+5)}{(x+2)(x+5)} = \dfrac{2x}{x+2}$

In the last line, dividing through by $(x + 5)$ is allowed because $x > 0$ so $x + 5 \neq 0$.

b Let $u = 2x$ and $v = x+2$

$\dfrac{du}{dx} = 2$ and $\dfrac{dv}{dx} = 1$

$f'(x) = \dfrac{2(x+2) - 2x}{(x+2)^2} = \dfrac{4}{(x+2)^2}$

Hence $f'(3) = \dfrac{4}{5^2} = \dfrac{4}{25}$

12 a $f(x) = \dfrac{2\cos 2x}{e^{2-x}}$

Let $u = 2\cos 2x$ and $v = e^{2-x}$

$\dfrac{du}{dx} = -4\sin 2x$ and $\dfrac{dv}{dx} = -e^{2-x}$

$$f'(x) = \frac{-4e^{2-x}\sin 2x - 2\cos 2x\,(-e^{2-x})}{(e^{2-x})^2}$$

$$= \frac{2e^{2-x}(\cos 2x - 2\sin 2x)}{(e^{2-x})^2}$$

At stationary points, $f'(x) = 0$

$\cos 2x - 2\sin 2x = 0$

$2\sin 2x = \cos 2x$

$\therefore\ \tan 2x = \frac{1}{2}$

b The range of f(x) is between the y-coordinate of B and the y-coordinate of the right endpoint of the interval.

$\tan 2x = \frac{1}{2} \Rightarrow 2x = 0.4636$ or 3.6052

$x = 0.2318$ or 1.8026

So the x-coordinate of B is 1.8026.

Range of f(x) is

$f(1.8026) \leqslant y < f(\pi)$

$-1.47 \leqslant y < 6.26$ (3 s.f.)