

Exercise 6D

1 a Let $y = x(1+3x)^5$

Let $u = x$ and $v = (1+3x)^5$

Then $\frac{du}{dx} = 1$ and $\frac{dv}{dx} = 3 \times 5(1+3x)^4$
(using the chain rule)

Using $\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$

$$\begin{aligned} \frac{dy}{dx} &= x \times 15(1+3x)^4 + (1+3x)^5 \times 1 \\ &= (1+3x)^4(15x+1+3x) \\ &= (1+3x)^4(1+18x) \end{aligned}$$

b Let $y = 2x(1+3x^2)^3$

Let $u = 2x$ and $v = (1+3x^2)^3$

Then $\frac{du}{dx} = 2$ and $\frac{dv}{dx} = 18x(1+3x^2)^2$

Using $\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$

$$\begin{aligned} \frac{dy}{dx} &= 2x \times 18x(1+3x^2)^2 + (1+3x^2)^3 \times 2 \\ &= (1+3x^2)^2(36x^2 + 2(1+3x^2)) \\ &= (1+3x^2)^2(42x^2 + 2) \\ &= 2(1+3x^2)^2(21x^2 + 1) \end{aligned}$$

c Let $y = x^3(2x+6)^4$

Let $u = x^3$ and $v = (2x+6)^4$

Then $\frac{du}{dx} = 3x^2$ and $\frac{dv}{dx} = 8(2x+6)^3$

Using $\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$

$$\begin{aligned} \frac{dy}{dx} &= x^3 \times 8(2x+6)^3 + (2x+6)^4 \times 3x^2 \\ &= x^2(2x+6)^3(8x+3(2x+6)) \\ &= x^2(2x+6)^3(14x+18) \\ &= x^2 \times 2^3(x+3)^3 \times 2(7x+9) \\ &= 16x^2(x+3)^3(7x+9) \end{aligned}$$

d Let $y = 3x^2(5x-1)^{-1}$

Let $u = 3x^2$ and $v = (5x-1)^{-1}$

Then $\frac{du}{dx} = 6x$ and $\frac{dv}{dx} = -5(5x-1)^{-2}$

Using $\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$

$$\begin{aligned} \frac{dy}{dx} &= 3x^2 \times (-5(5x-1)^{-2}) + (5x-1)^{-1} \times 6x \\ &= -15x^2(5x-1)^{-2} + 6x(5x-1)^{-1} \\ &= 3x(5x-1)^{-2}(-5x+2(5x-1)) \\ &= 3x(5x-2)(5x-1)^{-2} \end{aligned}$$

2 a Let $y = e^{-2x}(2x-1)^5$

Let $u = e^{-2x}$ and $v = (2x-1)^5$

Then $\frac{du}{dx} = -2e^{-2x}$ and $\frac{dv}{dx} = 10(2x-1)^4$

Using $\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$

$$\begin{aligned} \frac{dy}{dx} &= e^{-2x} \times 10(2x-1)^4 + (2x-1)^5(-2e^{-2x}) \\ &= e^{-2x}(2x-1)^4(10-2(2x-1)) \\ &= e^{-2x}(2x-1)^4(12-4x) \\ &= 4(3-x)(2x-1)^4 e^{-2x} \end{aligned}$$

2 b Let $y = \sin 2x \cos 3x$

Let $u = \sin 2x$ and $v = \cos 3x$

Then $\frac{du}{dx} = 2 \cos 2x$ and $\frac{dv}{dx} = -3 \sin 3x$

Using $\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$

$$\begin{aligned} \frac{dy}{dx} &= \sin 2x(-3 \sin 3x) + \cos 3x(2 \cos 2x) \\ &= 2 \cos 2x \cos 3x - 3 \sin 2x \sin 3x \end{aligned}$$

c Let $y = e^x \sin x$

Let $u = e^x$ and $v = \sin x$

Then $\frac{du}{dx} = e^x$ and $\frac{dv}{dx} = \cos x$

Using $\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$

$$\frac{dy}{dx} = e^x \cos x + \sin x e^x = e^x (\sin x + \cos x)$$

d Let $y = \sin(5x) \ln(\cos x)$

Let $u = \sin 5x$ and $v = \ln(\cos x)$

Then $\frac{du}{dx} = 5 \cos 5x$

and $\frac{dv}{dx} = (-\sin x) \times \frac{1}{\cos x} = -\tan x$

Using $\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$

$$\begin{aligned} \frac{dy}{dx} &= \sin(5x)(-\tan x) + 5 \cos(5x) \ln(\cos x) \\ &= 5 \cos 5x \ln(\cos x) - \tan x \sin 5x \end{aligned}$$

3 a $y = x^2(3x-1)^3$

Let $u = x^2$ and $v = (3x-1)^3$

Then $\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = 9(3x-1)^2$

Using the product rule $\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$

$$\begin{aligned} \frac{dy}{dx} &= x^2 \times 9(3x-1)^2 + (3x-1)^3 \times 2x \\ &= x(3x-1)^2 (9x + 2(3x-1)) \\ &= x(3x-1)^2 (15x-2) \quad (*) \end{aligned}$$

At the point (1, 8), $x = 1$.

Substituting $x = 1$ into expression (*):

$$\frac{dy}{dx} = 1 \times 2^2 \times 13 = 52$$

b $y = 3x(2x+1)^{\frac{1}{2}}$

Let $u = 3x$ and $v = (2x+1)^{\frac{1}{2}}$

Then $\frac{du}{dx} = 3$

and $\frac{dv}{dx} = 2 \times \frac{1}{2} (2x+1)^{-\frac{1}{2}} = (2x+1)^{-\frac{1}{2}}$

Using the product rule $\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$

$$\begin{aligned} \frac{dy}{dx} &= 3x(2x+1)^{-\frac{1}{2}} + 3(2x+1)^{\frac{1}{2}} \\ &= 3(2x+1)^{-\frac{1}{2}} (x + (2x+1)) \\ &= 3(3x+1)(2x+1)^{-\frac{1}{2}} \quad (*) \end{aligned}$$

At the point (4, 36), $x = 4$.

Substituting $x = 4$ into (*):

$$\frac{dy}{dx} = 3 \times 13 \times 9^{-\frac{1}{2}} = 3 \times 13 \times \frac{1}{3} = 13$$

3 c $y = (x-1)(2x+1)^{-1}$

Let $u = x-1$ and $v = (2x+1)^{-1}$

Then $\frac{du}{dx} = 1$ and $\frac{dv}{dx} = -2(2x+1)^{-2}$

Using the product rule $\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$

$$\begin{aligned} \frac{dy}{dx} &= (x-1)(-2(2x+1)^{-2}) + (2x+1)^{-1} \times 1 \\ &= (2x+1)^{-2}(-2(x-1) + (2x+1)) \\ &= 3(2x+1)^{-2} \quad (*) \end{aligned}$$

At the point $(2, \frac{1}{5})$, $x = 2$.

Substituting $x = 2$ into (*):

$$\frac{dy}{dx} = 3 \times 5^{-2} = \frac{3}{25}$$

4 $y = (x-2)^2(2x+3)$

Let $u = (x-2)^2$ and $v = (2x+3)$

Then $\frac{du}{dx} = 2(x-2)$ and $\frac{dv}{dx} = 2$

Using $\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$

$$\begin{aligned} \frac{dy}{dx} &= (x-2)^2 \times 2 + (2x+3) \times 2(x-2) \\ &= 2(x-2)(x-2+2x+3) \\ &= 2(x-2)(3x+1) \end{aligned}$$

At stationary points $\frac{dy}{dx} = 0$

$$2(x-2)(3x+1) = 0$$

$$(x-2) = 0 \text{ or } (3x+1) = 0$$

$$\therefore x = 2 \text{ or } -\frac{1}{3}$$

$$x = 2 \Rightarrow y = 0$$

$$x = -\frac{1}{3} \Rightarrow y = \left(-\frac{7}{3}\right)^2 \left(\frac{7}{3}\right) = \frac{343}{27}$$

So the stationary points are

$$(2, 0) \text{ and } \left(-\frac{1}{3}, \frac{343}{27}\right)$$

5 $y = \left(x - \frac{\pi}{2}\right)^5 \sin 2x$

Let $u = \left(x - \frac{\pi}{2}\right)^5$ and $v = \sin 2x$

$$\frac{du}{dx} = 5\left(x - \frac{\pi}{2}\right)^4 \text{ and } \frac{dv}{dx} = 2 \cos 2x$$

Using $\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$

$$\begin{aligned} \frac{dy}{dx} &= \left(x - \frac{\pi}{2}\right)^5 2 \cos 2x + \sin 2x \times 5\left(x - \frac{\pi}{2}\right)^4 \\ &= \left(x - \frac{\pi}{2}\right)^4 \left(2\left(x - \frac{\pi}{2}\right) \cos 2x + 5 \sin 2x\right) \end{aligned}$$

When $x = \frac{\pi}{4}$,

$$\begin{aligned} \frac{dy}{dx} &= \left(-\frac{\pi}{4}\right)^4 \left(2\left(-\frac{\pi}{4}\right) \cos \frac{\pi}{2} + 5 \sin \frac{\pi}{2}\right) \\ &= \frac{\pi^4}{256} (0 + 5) = \frac{5\pi^4}{256} \end{aligned}$$

6 $y = x^2 \cos(x^2)$

Let $u = x^2$ and $v = \cos(x^2)$

$$\frac{du}{dx} = 2x \quad \text{and} \quad \frac{dv}{dx} = -2x \sin(x^2)$$

Using $\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$

$$\begin{aligned} \frac{dy}{dx} &= x^2 (-2x \sin(x^2)) + \cos(x^2) \times 2x \\ &= 2x(\cos(x^2) - x^2 \sin(x^2)) \end{aligned}$$

When $x = \frac{\sqrt{\pi}}{2}$,

$$\begin{aligned} \frac{dy}{dx} &= \sqrt{\pi} \left(\cos \frac{\pi}{4} - \frac{\pi}{4} \sin \frac{\pi}{4} \right) \\ &= \sqrt{\pi} \left(\frac{1}{\sqrt{2}} - \frac{\pi}{4\sqrt{2}} \right) = \sqrt{\frac{\pi}{2}} \left(1 - \frac{\pi}{4} \right) \end{aligned}$$

Equation of tangent at $P \left(\frac{\sqrt{\pi}}{2}, \frac{\pi\sqrt{2}}{8} \right)$ is

$$y - \frac{\pi\sqrt{2}}{8} = \sqrt{\frac{\pi}{2}} \left(1 - \frac{\pi}{4} \right) \left(x - \frac{\sqrt{\pi}}{2} \right)$$

$$8y - \pi\sqrt{2} = 4\sqrt{2\pi} \left(1 - \frac{\pi}{4} \right) \left(x - \frac{\sqrt{\pi}}{2} \right)$$

$$8y - \pi\sqrt{2} = \sqrt{2\pi}(4 - \pi) \left(x - \frac{\sqrt{\pi}}{2} \right)$$

$$8y - \pi\sqrt{2} = \sqrt{2\pi}(4 - \pi)x - \frac{\pi\sqrt{2}}{2}(4 - \pi)$$

$$\sqrt{2\pi}(\pi - 4)x + 8y - \pi\sqrt{2} + \frac{\pi\sqrt{2}}{2}(4 - \pi) = 0$$

$$\sqrt{2\pi}(\pi - 4)x + 8y - \pi\sqrt{2} \left(\frac{\pi - 2}{2} \right) = 0$$

This is in the form $ax + by + c = 0$ with

$$a = \sqrt{2\pi}(\pi - 4), b = 8 \quad \text{and} \quad c = -\pi\sqrt{2} \left(\frac{\pi - 2}{2} \right)$$

7 $y = 3x^2(5x - 3)^3$

Let $u = 3x^2$ and $v = (5x - 3)^3$

$$\frac{du}{dx} = 6x \quad \text{and} \quad \frac{dv}{dx} = 15(5x - 3)^2$$

Using $\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$

$$\begin{aligned} \frac{dy}{dx} &= 3x^2 \times 15(5x - 3)^2 + 6x(5x - 3)^3 \\ &= 3x(5x - 3)^2 (15x + 2(5x - 3)) \\ &= 3x(5x - 3)^2 (25x - 6) \end{aligned}$$

Hence $A = 3, n = 2, B = 25$ and $C = -6$.

8 a $y = (x + 3)^2 e^{3x}$

Let $u = (x + 3)^2$ and $v = e^{3x}$

$$\frac{du}{dx} = 2(x + 3) \quad \text{and} \quad \frac{dv}{dx} = 3e^{3x}$$

Using $\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$

$$\begin{aligned} \frac{dy}{dx} &= (x + 3)^2 \times 3e^{3x} + e^{3x} \times 2(x + 3) \\ &= e^{3x}(x + 3)(3(x + 3) + 2) \\ &= e^{3x}(x + 3)(3x + 11) \end{aligned}$$

b When $x = 2$, $\frac{dy}{dx} = e^6 \times 5 \times 17 = 85e^6$

Hence the gradient at point C is $85e^6$.

9 a Let $y = (2 \sin x - 3 \cos x) \ln 3x$

Let $u = 2 \sin x - 3 \cos x$ and $v = \ln 3x$

Then $\frac{du}{dx} = 2 \cos x + 3 \sin x$ and $\frac{dv}{dx} = \frac{1}{x}$

Using $\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{2 \sin x - 3 \cos x}{x} \\ &\quad + (2 \cos x + 3 \sin x) \ln 3x \end{aligned}$$

9 b Let $y = x^4 e^{7x-3}$

Let $u = x^4$ and $v = e^{7x-3}$

$$\text{Then } \frac{du}{dx} = 4x^3 \text{ and } \frac{dv}{dx} = 7e^{7x-3}$$

$$\text{Using } \frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$\begin{aligned} \frac{dy}{dx} &= x^4 \times 7e^{7x-3} + 4x^3 e^{7x-3} \\ &= x^3 e^{7x-3} (7x+4) \end{aligned}$$

10 Let $y = x^5 \sqrt{10x+6}$

Let $u = x^5$ and $v = \sqrt{10x+6} = (10x+6)^{\frac{1}{2}}$

$$\text{Then } \frac{du}{dx} = 5x^4$$

$$\text{and } \frac{dv}{dx} = 10 \times \frac{1}{2} (10x+6)^{-\frac{1}{2}} = \frac{5}{\sqrt{10x+6}}$$

$$\text{Using } \frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{5x^5}{\sqrt{10x+6}} + 5x^4 \sqrt{10x+6}$$

$$\text{When } x=1, \frac{dy}{dx} = \frac{5}{\sqrt{16}} + 5\sqrt{16} = 21.25$$

Challenge

a Let $y = e^x \sin^2 x \cos x$

Let $p = e^x$ and $q = \sin^2 x \cos x$

$$\text{Then } y = pq, \quad \frac{dp}{dx} = e^x$$

$$\text{and } \frac{dy}{dx} = p \frac{dq}{dx} + q \frac{dp}{dx}$$

Let $u = \sin^2 x = (\sin x)^2$ and $v = \cos x$

Then $q = uv$,

$$\frac{du}{dx} = 2 \sin x \cos x \quad \text{and} \quad \frac{dv}{dx} = -\sin x$$

$$\text{Using } \frac{dq}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$\begin{aligned} \frac{dq}{dx} &= \sin^2 x (-\sin x) + \cos x (2 \sin x \cos x) \\ &= -\sin^3 x + 2 \sin x \cos^2 x \end{aligned}$$

$$\text{Using } \frac{dy}{dx} = p \frac{dq}{dx} + q \frac{dp}{dx}$$

$$\begin{aligned} \frac{dy}{dx} &= e^x (-\sin^3 x + 2 \sin x \cos^2 x) \\ &\quad + \sin^2 x \cos x \times e^x \end{aligned}$$

$$= -e^x \sin x (\sin^2 x - 2 \cos^2 x - \sin x \cos x)$$

Challenge

b Let $y = x(4x - 3)^6(1 - 4x)^9$

Let $p = x(4x - 3)^6$ and $q = (1 - 4x)^9$

Then $y = pq$, $\frac{dq}{dx} = -36(1 - 4x)^8$

and $\frac{dy}{dx} = p \frac{dq}{dx} + q \frac{dp}{dx}$

Let $u = x$ and $v = (4x - 3)^6$

Then $p = uv$,

$\frac{du}{dx} = 1$ and $\frac{dv}{dx} = 24(4x - 3)^5$

Using $\frac{dp}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$

$$\begin{aligned} \frac{dp}{dx} &= 24x(4x - 3)^5 + (4x - 3)^6 \\ &= (4x - 3)^5(24x + 4x - 3) \\ &= (4x - 3)^5(28x - 3) \end{aligned}$$

Using $\frac{dy}{dx} = p \frac{dq}{dx} + q \frac{dp}{dx}$

$$\begin{aligned} \frac{dy}{dx} &= x(4x - 3)^6(-36(1 - 4x)^8) \\ &\quad + (1 - 4x)^9(4x - 3)^5(28x - 3) \\ &= (4x - 3)^5(1 - 4x)^8 \\ &\quad \times (-36x(4x - 3) + (1 - 4x)(28x - 3)) \\ &= (4x - 3)^5(1 - 4x)^8 \\ &\quad \times (-144x^2 + 108x + 40x - 112x^2 - 3) \\ &= -(4x - 3)^5(1 - 4x)^8(256x^2 - 148x + 3) \end{aligned}$$