Exercise 6C

1 a $y=(1+2x)^4$

Let $u=1+2x$; then $y=u^4$

$\frac{du}{dx}=2$ and $\frac{dy}{du}=4u^3$

Using the chain rule,

$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=4u^3\times 2=8u^3=8(1+2x)^3$

b $y=(3-2x^2)^{-5}$

Let $u=3-2x^2$; then $y=u^{-5}$

$\frac{du}{dx}=-4x$ and $\frac{dy}{du}=-5u^{-6}$

Using the chain rule,

$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=(-5u^{-6})\times(-4x)=20xu^{-6}$

$=20x(3-2x^2)^{-6}$

c $y=(3+4x)^{\frac{1}{2}}$

Let $u=3+4x$; then $y=u^{\frac{1}{2}}$

$\frac{du}{dx}=4$ and $\frac{dy}{du}=\frac{1}{2}u^{-\frac{1}{2}}$

Using the chain rule,

$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=\frac{1}{2}u^{-\frac{1}{2}}\times 4$

$=2u^{-\frac{1}{2}}=2(3+4x)^{-\frac{1}{2}}$

d $y=(6x+x^2)^7$

Let $u=6x+x^2$; then $y=u^7$

$\frac{du}{dx}=6+2x$ and $\frac{dy}{du}=7u^6$

Using the chain rule,

$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=7u^6\times(6+2x)$

$=7(6+2x)(6x+x^2)^6$

1 e $y=\frac{1}{3+2x}=(3+2x)^{-1}$

Let $u=3+2x$; then $y=\frac{1}{u}=u^{-1}$

$\frac{du}{dx}=2$ and $\frac{dy}{du}=-u^{-2}$

Using the chain rule,

$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=-u^{-2}\times 2$

$=-2u^{-2}=\frac{-2}{(3+2x)^2}$

f $y=\sqrt{7-x}=(7-x)^{\frac{1}{2}}$

Let $u=7-x$; then $y=u^{\frac{1}{2}}$

$\frac{du}{dx}=-1$ and $\frac{dy}{du}=\frac{1}{2}u^{-\frac{1}{2}}$

Using the chain rule,

$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=\frac{1}{2}u^{-\frac{1}{2}}\times(-1)$

$=-\frac{1}{2}(7-x)^{-\frac{1}{2}}=-\frac{1}{2\sqrt{7-x}}$

g $y=4(2+8x)^4$

Let $u=2+8x$; then $y=4u^4$

$\frac{du}{dx}=8$ and $\frac{dy}{du}=16u^3$

Using the chain rule,

$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=16u^3\times 8=128(2+8x)^3$

h $y=3(8-x)^{-6}$

Let $u=8-x$; then $y=3u^{-6}$

$\frac{du}{dx}=-1$ and $\frac{dy}{du}=-18u^{-7}$

Using the chain rule,

$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=-18u^{-7}\times(-1)=18(8-x)^{-7}$

2 a $y = e^{\cos x}$

Let $u = \cos x$; then $y = e^u$

$$\frac{du}{dx} = -\sin x \quad \text{and} \quad \frac{dy}{du} = e^u$$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= e^u \times (-\sin x) = -\sin x \, e^{\cos x}$$

b $y = \cos(2x-1)$

Let $u = 2x-1$; then $y = \cos u$

$$\frac{du}{dx} = 2 \quad \text{and} \quad \frac{dy}{du} = -\sin u$$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -\sin u \times 2 = -2\sin(2x-1)$$

c $y = \sqrt{\ln x}$

Let $u = \ln x$; then $y = \sqrt{u} = u^{\frac{1}{2}}$

$$\frac{du}{dx} = -\frac{1}{x} \quad \text{and} \quad \frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{2}u^{-\frac{1}{2}} \times \left(-\frac{1}{x}\right)$$
$$= \frac{1}{2xu^{\frac{1}{2}}} = \frac{1}{2x\sqrt{\ln x}}$$

d $y = (\sin x + \cos x)^5$

Let $u = \sin x + \cos x$; then $y = u^5$

$$\frac{du}{dx} = \cos x - \sin x \quad \text{and} \quad \frac{dy}{du} = 5u^4$$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 5u^4 \times (\cos x - \sin x)$$
$$= 5(\cos x - \sin x)(\sin x + \cos x)^4$$

2 e $y = \sin(3x^2 - 2x + 1)$

Let $u = 3x^2 - 2x + 1$; then $y = \sin u$

$$\frac{du}{dx} = 6x - 2 \quad \text{and} \quad \frac{dy}{du} = \cos u$$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \cos u \times (6x-2)$$
$$= (6x-2)\cos(3x^2 - 2x + 1)$$

f $y = \ln(\sin x)$

Let $u = \sin x$; then $y = \ln u$

$$\frac{du}{dx} = \cos x \quad \text{and} \quad \frac{dy}{du} = \frac{1}{u}$$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{u} \times \cos x = \frac{\cos x}{\sin x} = \cot x$$

g $y = 2e^{\cos 4x}$

Let $u = \cos 4x$; then $y = 2e^u$

$$\frac{du}{dx} = -4\sin 4x \quad \text{and} \quad \frac{dy}{du} = 2e^u$$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 2e^u \times (-4\sin 4x)$$
$$= -8\sin 4x \, e^{\cos x}$$

2 h $y = \cos(e^{2x} + 3)$

Let $u = e^{2x} + 3$; then $y = \cos u$

$$\frac{du}{dx} = 2e^{2x} \quad \text{and} \quad \frac{dy}{du} = -\sin u$$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -\sin u \times 2e^{2x}$$
$$= -2e^{2x}\sin(e^{2x} + 3)$$

3 $y = \dfrac{1}{(4x+1)^2}$

Let $u = 4x+1$; then $y = u^{-2}$

$\dfrac{du}{dx} = 4$ and $\dfrac{dy}{du} = -2u^{-3}$

$\therefore \dfrac{dy}{dx} = -8u^{-3} = \dfrac{-8}{(4x+1)^3}$

When $x = \frac{1}{4}$,

$\dfrac{dy}{dx} = \dfrac{-8}{\left(4\times\frac{1}{4}+1\right)^3} = \dfrac{-8}{2^3} = -1$

4 $y = (5-2x)^3$

Let $u = 5-2x$; then $y = u^3$

$\dfrac{du}{dx} = -2$ and $\dfrac{dy}{du} = 3u^2$

Using the chain rule,

$\dfrac{dy}{dx} = \dfrac{dy}{du}\times\dfrac{du}{dx} = 3u^2\times(-2) = -6(5-2x)^2$

When $x = 1$,

$y = 3^3 = 27$ and $\dfrac{dy}{dx} = -6\times 3^2 = -54$

Equation of tangent at point P (1, 27) is

$y - 27 = -54(x-1)$

or $y = -54x + 81$

5 $y = (1+\ln 4x)^{\frac{3}{2}}$

Let $u = 1+\ln 4x$; then $y = u^{\frac{3}{2}}$

$\dfrac{du}{dx} = \dfrac{1}{x}$ and $\dfrac{dy}{du} = \dfrac{3}{2}u^{\frac{1}{2}}$

Using the chain rule,

$\dfrac{dy}{dx} = \dfrac{dy}{du}\times\dfrac{du}{dx} = \dfrac{3}{2}u^{\frac{1}{2}}\times\dfrac{1}{x} = \dfrac{3}{2x}\sqrt{1+\ln 4x}$

When $x = \frac{1}{4}e^3$,

$\dfrac{dy}{dx} = \dfrac{3}{\frac{1}{2}e^3}\sqrt{1+\ln e^3} = \dfrac{6}{e^3}\sqrt{1+3} = 12e^{-3}$

6 a $x = y^2 + y$

$\dfrac{dx}{dy} = 2y+1$

$\therefore \dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}} = \dfrac{1}{2y+1}$

b $x = e^y + 4y$

$\dfrac{dx}{dy} = e^y + 4$

$\therefore \dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}} = \dfrac{1}{e^y+4}$

c $x = \sin 2y$

$\dfrac{dx}{dy} = 2\cos 2y$

$\therefore \dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}} = \dfrac{1}{2\cos 2y} = \dfrac{1}{2}\sec 2y$

d $4x = \ln y + y^3$

$x = \dfrac{1}{4}\ln y + \dfrac{1}{4}y^3$

$\dfrac{dx}{dy} = \dfrac{1}{4y} + \dfrac{3}{4}y^2 = \dfrac{1+3y^3}{4y}$

$\therefore \dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}} = \dfrac{4y}{1+3y^3}$

7 $x = 3y^2 - 2y$

$\dfrac{dx}{dy} = 6y-2$

$\dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}} = \dfrac{1}{6y-2}$

At (8, 2) the value of y is 2.

$\therefore \dfrac{dy}{dx} = \dfrac{1}{12-2} = \dfrac{1}{10}$

8 $x = y^{\frac{1}{2}} + y^{-\frac{1}{2}}$

$\frac{dx}{dy} = \frac{1}{2}y^{-\frac{1}{2}} - \frac{1}{2}y^{-\frac{3}{2}}$

$\frac{dy}{dx} = \frac{1}{\frac{1}{2}y^{-\frac{1}{2}} - \frac{1}{2}y^{-\frac{3}{2}}}$

At the point $\left(\frac{5}{2}, 4\right)$ the value of y is 4.

$\therefore \frac{dy}{dx} = \frac{1}{\frac{1}{2}(4)^{-\frac{1}{2}} - \frac{1}{2}(4)^{-\frac{3}{2}}} = \frac{1}{\frac{1}{2}\times\frac{1}{2} - \frac{1}{2}\times\frac{1}{8}}$

$= \frac{1}{\frac{1}{4} - \frac{1}{16}} = \frac{1}{\frac{3}{16}} = \frac{16}{3}$

9 a $x = e^y$

$\Rightarrow \frac{dx}{dy} = e^y$

b $y = \ln x \Rightarrow e^y = x$

From part **a**, $\frac{dx}{dy} = e^y$

$\therefore \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{e^y} = \frac{1}{x}$

10 a $x = 4\cos 2y$

When $x = 2$, $\cos 2y = \frac{1}{2}$

So $2y = \frac{\pi}{3}, \frac{5\pi}{3}, \ldots$

$y = \frac{\pi}{6}, \frac{5\pi}{6}, \ldots$

Hence $Q\left(2, \frac{\pi}{6}\right)$ lies on C.

b $\frac{dx}{dy} = -8\sin 2y \Rightarrow \frac{dy}{dx} = -\frac{1}{8\sin 2y}$

At Q, $y = \frac{\pi}{6}$

so $\frac{dy}{dx} = -\frac{1}{8\sin\frac{\pi}{3}} = -\frac{1}{8\times\frac{\sqrt{3}}{2}} = -\frac{1}{4\sqrt{3}}$

c Equation of normal to C at Q is

$y - \frac{\pi}{6} = 4\sqrt{3}(x-2)$

or $4\sqrt{3}x - y - 8\sqrt{3} + \frac{\pi}{6} = 0$

11 a $y = \sin^2 3x = (\sin 3x)^2$

Let $u = \sin 3x$; then $y = u^2$

$\frac{du}{dx} = 3\cos 3x$ and $\frac{dy}{du} = 2u$

Using the chain rule,

$\frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx} = 2u \times 3\cos 3x$

$= 6\sin 3x\cos 3x$

b $y = e^{(x+1)^2}$

Let $u = (x+1)^2$; then $y = e^u$

$\frac{du}{dx} = 2(x+1)$ and $\frac{dy}{du} = e^u$

Using the chain rule,

$\frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx} = e^u \times 2(x+1) = 2(x+1)e^{(x+1)^2}$

c $y = \ln(\cos x)^2$

Let $u = \cos x$; then $y = \ln u^2 = 2\ln u$

$\frac{du}{dx} = -\sin x$ and $\frac{dy}{du} = \frac{2}{u}$

Using the chain rule,

$\frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx} = \frac{2}{u}\times(-\sin x)$

$= -2\frac{\sin x}{\cos x} = -2\tan x$

d $y = \frac{1}{3+\cos 2x}$

Let $u = 3 + \cos 2x$; then $y = \frac{1}{u} = u^{-1}$

$\frac{du}{dx} = -2\sin 2x$ and $\frac{dy}{du} = -u^{-2} = -\frac{1}{u^2}$

Using the chain rule,

$\frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx} = -\frac{1}{u^2}\times(-2\sin 2x)$

$= \frac{2\sin 2x}{(3+\cos 2x)^2}$

11 e $y = \sin\left(\frac{1}{x}\right)$

Let $u = \frac{1}{x}$; then $y = \sin u$

$\frac{du}{dx} = -\frac{1}{x^2}$ and $\frac{dy}{du} = \cos u$

Using the chain rule,

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \cos u \times \left(-\frac{1}{x^2}\right)$

$= -\frac{1}{x^2}\cos\left(\frac{1}{x}\right)$

12 $y = \frac{4}{(2-4x)^2}$

Let $u = 2-4x$; then $y = \frac{4}{u^2} = 4u^{-2}$

$\frac{du}{dx} = -4$ and $\frac{dy}{du} = -8u^{-3} = -\frac{8}{u^3}$

Using the chain rule,

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -\frac{8}{u^3} \times (-4) = \frac{32}{(2-4x)^3}$

When $x = 3$, $y = \frac{4}{(-10)^2} = 0.04$

and $\frac{dy}{dx} = \frac{32}{(-10)^3} = -0.032$

Equation of normal at A is

$y - 0.04 = \frac{1}{0.032}(x-3)$

Multiplying through by 100 and rearranging gives

$100y - 4 = 3125x - 9375$

$3125x - 100y - 9371 = 0$

13 $y = 3^{x^3}$

Let $u = x^3$; then $y = 3^u$

$\frac{du}{dx} = 3x^2$ and $\frac{dy}{du} = 3^u \ln 3$

Using the chain rule,

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 3^u \ln 3 \times 3x^2 = 3x^2 3^{x^3} \ln 3$

When $x = 1$, $\frac{dy}{dx} = 3 \times 1^2 \times 3^{1^3} \times \ln 3 = 9\ln 3$

Challenge

a $y = \sqrt{\sin\sqrt{x}}$

Let $u = \sqrt{x} = x^{\frac{1}{2}}$; then $y = \sqrt{\sin u}$

$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$

Let $v = \sin u$; then $y = \sqrt{v} = v^{\frac{1}{2}}$

$\frac{dv}{du} = \cos u$ and $\frac{dy}{dv} = \frac{1}{2}v^{-\frac{1}{2}}$

Using the chain rule,

$\frac{dy}{du} = \frac{dy}{dv} \times \frac{dv}{du} = \frac{1}{2}v^{-\frac{1}{2}} \times \cos u$

$= \frac{\cos u}{2y} = \frac{\cos\sqrt{x}}{2\sqrt{\sin\sqrt{x}}}$

Using the chain rule again,

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{\cos\sqrt{x}}{2\sqrt{\sin\sqrt{x}}} \times \frac{1}{2}x^{-\frac{1}{2}}$

$= \frac{\cos\sqrt{x}}{4\sqrt{x\sin\sqrt{x}}}$

b $\ln y = \sin^3(3x+4)$

Hence $y = e^{\sin^3(3x+4)}$

Let $u = \sin(3x+4)$; then $y = e^{u^3}$

$\frac{du}{dx} = 3\cos(3x+4)$

Let $v = u^3$; then $y = e^v$

$\frac{dv}{du} = 3u^2$ and $\frac{dy}{dv} = e^v$

Using the chain rule,

$\frac{dy}{du} = \frac{dy}{dv} \times \frac{dv}{du} = e^v \times 3u^2 = 3u^2 e^{u^3}$

Using the chain rule again,

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 3u^2 e^{u^3} \times 3\cos(3x+4)$

$= 3\sin^2(3x+4)e^{\sin^3(3x+4)} \times 3\cos(3x+4)$

$= 9e^{\sin^3(3x+4)}\cos(3x+4)\sin^2(3x+4)$