Exercise 6B

1 a $y = 4e^{7x}$

$\frac{dy}{dx} = 4 \times 7e^{7x} = 28e^{7x}$

b $y = 3^x$

$y = e^{\ln(3^x)} = e^{x\ln 3} = e^{(\ln 3)x}$

$\frac{dy}{dx} = \ln 3\, e^{(\ln 3)x} = \ln 3\, e^{\ln(3^x)} = 3^x \ln 3$

c $y = \left(\frac{1}{2}\right)^x$

Using the result $y = a^{kx} \Rightarrow \frac{dy}{dx} = a^{kx} k \ln a$

with $a = \frac{1}{2}$ and $k = 1$:

$\frac{dy}{dx} = \left(\frac{1}{2}\right)^x \ln \frac{1}{2}$

d $y = \ln 5x$

$y = \ln 5 + \ln x$

$\frac{dy}{dx} = 0 + \frac{1}{x} = \frac{1}{x}$

e $y = 4\left(\frac{1}{3}\right)^x$

Using the result $y = a^{kx} \Rightarrow \frac{dy}{dx} = a^{kx} k \ln a$

with $a = \frac{1}{3}$ and $k = 1$:

$\frac{dy}{dx} = 4\left(\frac{1}{3}\right)^x \ln \frac{1}{3}$

f $y = \ln(2x^3)$

$y = \ln 2 + \ln(x^3) = \ln 2 + 3\ln x$

$\frac{dy}{dx} = 0 + 3 \times \frac{1}{x} = \frac{3}{x}$

g $y = e^{3x} - e^{-3x}$

$\frac{dy}{dx} = 3e^{3x} - (-3e^{-3x})$

$= 3e^{3x} + 3e^{-3x}$

h $y = \frac{(1+e^x)^2}{e^x}$

$y = \frac{1 + 2e^x + (e^x)^2}{e^x} = e^{-x} + 2 + e^x$

$\frac{dy}{dx} = -e^{-x} + 0 + e^x = -e^{-x} + e^x$

2 a $f(x) = 3^{4x}$

$f(x) = e^{\ln(3^{4x})} = e^{4x\ln 3} = e^{(4\ln 3)x}$

$f'(x) = (4\ln 3)e^{(4\ln 3)x} = 4\ln 3\, e^{4x\ln 3}$

$= 4\ln 3\, e^{\ln 3^{4x}} = 3^{4x} 4\ln 3$

b $f(x) = \left(\frac{3}{2}\right)^{2x}$

Using the result $y = a^{kx} \Rightarrow \frac{dy}{dx} = a^{kx} k \ln a$

with $a = \frac{3}{2}$ and $k = 2$:

$f'(x) = \left(\frac{3}{2}\right)^{2x} 2\ln \frac{3}{2}$

c $f(x) = 2^{4x} + 4^{2x}$

Using the result $y = a^{kx} \Rightarrow \frac{dy}{dx} = a^{kx} k \ln a$

for each term:

$f'(x) = 2^{4x} 4\ln 2 + 4^{2x} 2\ln 4$

Alternatively,

$f(x) = 2^{4x} + (2^2)^{2x} = 2^{4x} + 2^{4x} = 2 \times 2^{4x}$

$f'(x) = 2 \times 2^{4x} 4\ln 2$

d $f(x) = \frac{2^{7x} + 8^x}{4^{2x}}$

$f(x) = \frac{2^{7x} + 2^{3x}}{2^{4x}} = 2^{3x} + 2^{-x}$

$f'(x) = 2^{3x} 3\ln 2 + 2^{-x}(-1)\ln 2$

$= 2^{3x} 3\ln 2 - 2^{-x} \ln 2$

3 $y = (e^{2x} - e^{-2x})^2 = e^{4x} - 2 + e^{-4x}$

$\frac{dy}{dx} = 4e^{4x} - 4e^{-4x} = 4(e^{4x} - e^{-4x})$

Where $x = \ln 3$:

$\frac{dy}{dx} = 4(e^{4\ln 3} - e^{-4\ln 3}) = 4\left(e^{\ln 3^4} - e^{\ln 3^{-4}}\right)$

$= 4\left(3^4 - 3^{-4}\right) = 4\left(81 - \frac{1}{81}\right)$

≈ 323.95

4 $y = 2^x + 2^{-x}$

$\frac{dy}{dx} = 2^x \ln 2 - 2^{-x} \ln 2$

When $x = 2$, $\frac{dy}{dx} = 4 \ln 2 - \frac{1}{4} \ln 2 = \frac{15}{4} \ln 2$

$\therefore$ the equation of the tangent at $\left(2, \frac{17}{4}\right)$ is

$y - \frac{17}{4} = \frac{15}{4} \ln 2(x - 2)$

or $4y = (15 \ln 2)x + (17 - 30 \ln 2)$

5 $y = e^{2x} - \ln x$

$\frac{dy}{dx} = 2e^{2x} - \frac{1}{x}$

When $x = 1$, $y = e^2$ and $\frac{dy}{dx} = 2e^2 - 1$

Equation of tangent at $(1, e^2)$ is

$y - e^2 = (2e^2 - 1)(x - 1)$

Rearranging gives

$y = (2e^2 - 1)x - 2e^2 + 1 + e^2$

or $y = (2e^2 - 1)x - e^2 + 1$

6 $R = 200 \times 0.9^t$

$\frac{dR}{dt} = 200 \times (0.9)^t \ln 0.9 = 100 \ln 0.9 \times (0.9)^t$

When $t = 8$:

$\frac{dR}{dt} = 200 \ln 0.9 \times 0.9^8 = -9.07$ (3 s.f.)

7 a When $t = 0$, $P = 37\,000$

So $37\,000 = P_0 k^0 = P_0$

$P_0 = 37\,000$

When $t = 100$, $P = 109\,000$

So $109\,000 = 37\,000 k^{100}$

$\frac{109}{37} = k^{100}$

and hence $k = \left(\frac{109}{37}\right)^{\frac{1}{100}}$

$= 1.01086287...$

$= 1.01$ (3 s.f.)

7 b $P = P_0 k^t \Rightarrow \frac{dP}{dt} = P_0 k^t \ln k$

With $P_0 = 37\,000$, $k = 1.01086...$, $t = 100$:

$\frac{dP}{dt} = 37\,000 \times 1.0108629^{100} \times \ln 1.0108629$

$= 1178$

c The rate of change of population in the year 2000.

8 The student has treated $\ln kx$ as if it were e^{kx} – they applied the incorrect differentiation formula.

The correct derivative is $\frac{1}{x}$

9 Let $y = a^{kx}$

Then $y = e^{\ln a^{kx}} = e^{kx \ln a} = e^{(k \ln a)x}$

$\frac{dy}{dx} = (k \ln a)e^{(k \ln a)x} = k \ln a \, e^{kx \ln a}$

$= k \ln a \, e^{\ln a^{kx}} = a^{kx} k \ln a$

10 a $f(x) = e^{2x} - \ln(x^2) + 4 = e^{2x} - 2\ln x + 4$

$f'(x) = 2e^{2x} - \frac{2}{x}$

b At P,

$f'(x) = 2$ and $x = a$

so $2e^{2a} - \frac{2}{a} = 2$

$e^{2a} - \frac{1}{a} - 1 = 0$

$ae^{2a} - 1 - a = 0$

$\therefore a(e^{2a} - 1) = 1$

11 a $y = 5\sin 3x + 2\cos 3x$

When $x = 0$,

$y = 5\sin 0 + 2\cos 0 = 0 + 2 = 2$

Hence P (0, 2) lies on the curve.

b $\frac{dy}{dx} = 15\cos 3x - 6\sin 3x$

When $x = 0$, $\frac{dy}{dx} = 15\cos 0 - 6\sin 0 = 15$

Equation of normal at P is

$y - 2 = -\frac{1}{15}(x - 0)$

or $y = -\frac{1}{15}x + 2$

12 $y = 2\times 3^{4x}$

$\frac{dy}{dx} = 2\times 3^{4x}\, 4\ln 3 = 8\ln 3\times 3^{4x}$

When $x=1$, $y = 2\times 81 = 162$

and $\frac{dy}{dx} = 8\ln 3\times 3^4 = 648\ln 3$

Equation of normal at P is

$y - 162 = -\frac{1}{648\ln 3}(x-1)$

or $y = -\frac{1}{648\ln 3}x + \frac{1}{648\ln 3} + 162$

Challenge

$y = e^{4x} - 5x$

$\frac{dy}{dx} = 4e^{4x} - 5$

Lines parallel to $y = 3x+4$ have gradient 3.

$\frac{dy}{dx} = 3 \Rightarrow 4e^{4x} - 5 = 3$

$e^{4x} = 2$

$4x = \ln 2$

$x = \frac{\ln 2}{4}$

When $x = \frac{\ln 2}{4}$, $y = e^{\ln 2} - 5\frac{\ln 2}{4} = 2 - 5\frac{\ln 2}{4}$

Equation of tangent at this point is

$y - \left(2 - 5\frac{\ln 2}{4}\right) = 3\left(x - \frac{\ln 2}{4}\right)$

$y = 3x - 3\frac{\ln 2}{4} + 2 - 5\frac{\ln 2}{4}$

$y = 3x - 2\ln 2 + 2$