

Exercise 4E

1 $5 \sin \theta + 12 \cos \theta \equiv R \sin \theta \cos \alpha + R \cos \theta \sin \alpha$

Comparing $\sin \theta$: $R \cos \alpha = 5$

Comparing $\cos \theta$: $R \sin \alpha = 12$

Divide the equations:

$$\frac{\sin \alpha}{\cos \alpha} = \frac{12}{5} \Rightarrow \tan \alpha = \frac{12}{5}$$

Square and add the equations:

$$R^2 \cos^2 \alpha + R^2 \sin^2 \alpha = 5^2 + 12^2$$

$$R^2 (\cos^2 \alpha + \sin^2 \alpha) = 13^2$$

$$R = 13$$

$$\text{since } \cos^2 \alpha + \sin^2 \alpha \equiv 1$$

2 $\sqrt{3} \sin \theta + \sqrt{6} \cos \theta$
 $\equiv 3 \cos \theta \cos \alpha + 3 \sin \theta \sin \alpha$

Comparing $\sin \theta$: $\sqrt{3} = 3 \sin \alpha$ (1)

Comparing $\cos \theta$: $\sqrt{6} = 3 \cos \alpha$ (2)

Divide (1) by (2):

$$\tan \alpha = \frac{\sqrt{3}}{\sqrt{6}} = \frac{1}{\sqrt{2}}$$

So $\alpha = 35.3^\circ$

3 $2 \sin \theta - \sqrt{5} \cos \theta$
 $\equiv -3 \cos \theta \cos \alpha + 3 \sin \theta \sin \alpha$

Comparing $\sin \theta$: $2 = 3 \sin \alpha$ (1)

Comparing $\cos \theta$: $+\sqrt{5} = +3 \cos \alpha$ (2)

Divide (1) by (2):

$$\tan \alpha = \frac{2}{\sqrt{5}}$$

So $\alpha = 41.8^\circ$

4 a Let $\cos \theta - \sqrt{3} \sin \theta \equiv R \cos(\theta + \alpha)$
 $\equiv R \cos \theta \cos \alpha - R \sin \theta \sin \alpha$

Compare $\cos \theta$: $R \cos \alpha = 1$ (1)

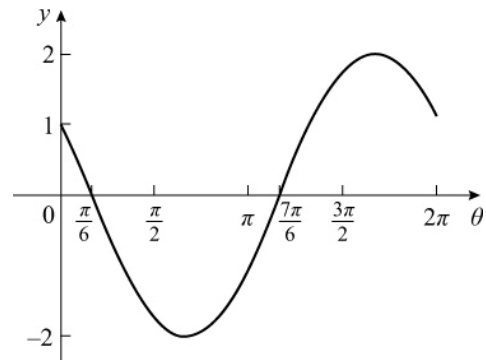
Compare $\sin \theta$: $R \sin \alpha = \sqrt{3}$ (2)

Divide (2) by (1): $\tan \alpha = \sqrt{3} \Rightarrow \alpha = \frac{\pi}{3}$

Square and add: $R^2 = 1 + 3 = 4 \Rightarrow R = 2$

So $\cos \theta - \sqrt{3} \sin \theta \equiv 2 \cos\left(\theta + \frac{\pi}{3}\right)$

- b This is the graph of $y = \cos \theta$, translated by $\frac{\pi}{3}$ to the left and then stretched in the y direction by scale factor 2.



Meets y -axis at $(0, 1)$

Meets x -axis at $\left(\frac{\pi}{6}, 0\right)$, $\left(\frac{7\pi}{6}, 0\right)$

5 a Let $7 \cos \theta - 24 \sin \theta \equiv R \cos(\theta + \alpha)$
 $\equiv R \cos \theta \cos \alpha - R \sin \theta \sin \alpha$

Compare $\cos \theta$: $R \cos \alpha = 7$ (1)

Compare $\sin \theta$: $R \sin \alpha = 24$ (2)

Divide (2) by (1): $\tan \alpha = \frac{24}{7}$

$$\Rightarrow \alpha = 73.7^\circ$$

Square and add: $R^2 = 24^2 + 7^2$

$$\Rightarrow R = 25$$

So $7 \cos \theta - 24 \sin \theta \equiv 25 \cos(\theta + 73.7^\circ)$

- b Graph meets y -axis where $\theta = 0$,
 i.e. $y = 7 \cos 0^\circ - 24 \sin 0^\circ = 7$
 so coordinates of P are $(0, 7)$

- c Maximum value of $25 \cos(\theta + 73.7^\circ)$ is
 when $\cos(\theta + 73.7^\circ) = 1$

So maximum is 25

Minimum value is $25(-1) = -25$

5 d i The line $y = 15$ will meet the graph twice in $0 < \theta < 360^\circ$, so there are 2 solutions.

ii As the maximum value is 25 it can never be 26, so there are 0 solutions.

iii As -25 is a minimum, line $y = -25$ only meets curve once, so only 1 solution.

6 a Let $\sin \theta + 3 \cos \theta \equiv R \sin(\theta + \alpha)$
 $\equiv R \sin \theta \cos \alpha + R \cos \theta \sin \alpha$

Comparing $\sin \theta$: $R \cos \alpha = 1$ (1)

Comparing $\cos \theta$: $R \sin \alpha = 3$ (2)

Divide (2) by (1)

$$\frac{R \sin \alpha}{R \cos \alpha} = \tan \alpha = 3$$

So $\alpha = 71.56^\circ$ (2 d.p.)

$$R^2 \cos^2 \alpha + R^2 \sin^2 \alpha = 1^2 + 3^2$$

$$R^2 (\cos^2 \alpha + \sin^2 \alpha) = 10$$

$$R^2 = 10$$

$$R = \sqrt{10}, \alpha = 71.6^\circ$$

b Use the value of α to 2 d.p. in calculating values of θ to avoid rounding errors

$$\sqrt{10} \sin(\theta + 71.56^\circ) = 2$$

$$\sin(\theta + 71.56^\circ) = \frac{2}{\sqrt{10}}$$

$$\sin^{-1}\left(\frac{2}{\sqrt{10}}\right) = 39.23^\circ \text{ (2 d.p.)}$$

As $0 \leq \theta < 360^\circ$, the interval for

$(\theta + 71.56^\circ)$ is

$$71.56^\circ \leq \theta + 71.56^\circ < 431.56^\circ$$

$$\text{So } \theta + 71.56^\circ = 180^\circ - 39.23^\circ,$$

$$\text{and } \theta + 71.56^\circ = 360^\circ + 39.23^\circ$$

$$\theta + 71.56^\circ = 140.77^\circ, 399.23^\circ$$

$$\theta = 69.2^\circ, 327.7^\circ$$

7 a Set $\cos 2\theta - 2 \sin 2\theta \equiv R \cos(2\theta + \alpha)$

$$\begin{aligned} \cos 2\theta - 2 \sin 2\theta \\ \equiv R \cos 2\theta \cos \alpha - R \sin 2\theta \sin \alpha \end{aligned}$$

Comparing $\sin 2\theta$: $R \sin \alpha = 2$ (1)

Comparing $\cos 2\theta$: $R \cos \alpha = 1$ (2)

Divide (1) by (2)

$$\frac{R \sin \alpha}{R \cos \alpha} = \tan \alpha = 2$$

So $\alpha = 1.107$

$$R^2 \cos^2 \alpha + R^2 \sin^2 \alpha = 1^2 + 2^2$$

$$R^2 (\cos^2 \alpha + \sin^2 \alpha) = 5$$

$$R = \sqrt{5}$$

So $\cos 2\theta - 2 \sin 2\theta = \sqrt{5} \cos(2\theta + 1.107)$

b $\sqrt{5} \cos(2\theta + 1.107) = -1.5$

$$\cos(2\theta + 1.107) = -\frac{1.5}{\sqrt{5}}$$

$$\cos^{-1}\left(-\frac{1.5}{\sqrt{5}}\right) = 2.306 \text{ (3 d.p.)}$$

As $0 \leq \theta < \pi$, the interval for

$(2\theta + 1.107)$ is

$$1.107 \leq 2\theta + 1.107 < 2\pi + 1.107$$

So $2\theta + 1.107 = 2.306, 2\pi - 2.306$

$$2\theta + 1.107 = 2.306, 3.977$$

$$\theta = 0.60, 1.44$$

- 8 a Write $6\sin x + 8\cos x$ in the form $R\sin(x + \alpha)$, where $R > 0$, $0 < \alpha < 90^\circ$

$$\begin{aligned} \text{So } 6\sin x + 8\cos x & \\ & \equiv R\sin x \cos \alpha + R\cos x \sin \alpha \end{aligned}$$

$$\text{Compare } \sin x: R\cos \alpha = 6 \quad (1)$$

$$\text{Compare } \cos x: R\sin \alpha = 8 \quad (2)$$

$$\text{Divide (2) by (1): } \tan \alpha = \frac{4}{3}$$

$$\Rightarrow \alpha = 53.13^\circ \text{ (2 d.p.)}$$

$$R^2 = 6^2 + 8^2 \Rightarrow R = 10$$

$$\text{So } 6\sin x + 8\cos x \equiv 10\sin(x + 53.13^\circ)$$

$$\text{Solve } 10\sin(x + 53.13^\circ) = 5\sqrt{3},$$

in the interval $0 \leq x \leq 360^\circ$

$$\text{so } \sin(x + 53.13^\circ) = \frac{\sqrt{3}}{2}$$

$$\Rightarrow x + 53.13^\circ = 60^\circ, 120^\circ$$

$$\Rightarrow x = 6.9^\circ, 66.9^\circ$$

- b Let $2\cos 3\theta - 3\sin 3\theta \equiv R\cos(3\theta + \alpha)$
 $\equiv R\cos 3\theta \cos \alpha - R\sin 3\theta \sin \alpha$

$$\text{Compare } \cos 3\theta: R\cos \alpha = 2 \quad (1)$$

$$\text{Compare } \sin 3\theta: R\sin \alpha = 3 \quad (2)$$

$$\text{Divide (2) by (1): } \tan \alpha = \frac{3}{2}$$

$$\Rightarrow \alpha = 56.31^\circ \text{ (2 d.p.)}$$

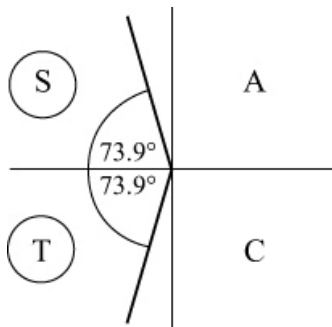
$$R^2 = 2^2 + 3^2 \Rightarrow R = \sqrt{13}$$

$$\text{Solve } \sqrt{13}\cos(3\theta + 56.31^\circ) = -1,$$

in the interval $0 \leq \theta \leq 90^\circ$

$$\text{so } \cos(3\theta + 56.31^\circ) = -\frac{1}{\sqrt{13}}$$

for $56.31^\circ \leq 3\theta + 56.31^\circ \leq 326.31^\circ$



$$\Rightarrow 3\theta + 56.31^\circ = 106.10^\circ, 253.90^\circ$$

$$\Rightarrow 3\theta = 49.8^\circ, 197.6^\circ$$

$$\Rightarrow \theta = 16.6^\circ, 65.9^\circ$$

- c Let $8\cos \theta + 15\sin \theta \equiv R\cos(\theta - \alpha)$
 $\equiv R\sin \theta \cos \alpha + R\cos \theta \sin \alpha$

$$\text{Compare } \cos \theta: R\cos \alpha = 8 \quad (1)$$

$$\text{Compare } \sin \theta: R\sin \alpha = 15 \quad (2)$$

$$\text{Divide (2) by (1): } \tan \alpha = \frac{15}{8}$$

$$\Rightarrow \alpha = 61.93^\circ \text{ (2 d.p.)}$$

$$R^2 = 8^2 + 15^2 \Rightarrow R = 17$$

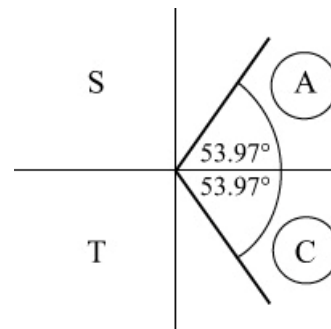
$$\text{Solve } 17\cos(\theta - 61.93^\circ) = 10,$$

in the interval $0 \leq \theta \leq 360^\circ$

$$\text{So } \cos(\theta - 61.93^\circ) = \frac{10}{17},$$

$$-61.93^\circ \leq \theta - 61.93^\circ \leq 298.07^\circ$$

$$\cos^{-1}\left(\frac{10}{17}\right) = 53.97^\circ \text{ (2 d.p.)}$$



$$\text{So } \theta - 61.93^\circ = -53.97^\circ, +53.97^\circ$$

$$\Rightarrow \theta = 8.0^\circ, 115.9^\circ$$

$$\begin{aligned}
 8 \text{ d} \quad & \text{Let } 5 \sin \frac{x}{2} - 12 \cos \frac{x}{2} \\
 & \equiv R \sin \frac{x}{2} - \alpha \\
 & \equiv R \sin \frac{x}{2} \cos \alpha - R \cos \frac{x}{2} \sin \alpha
 \end{aligned}$$

$$\text{Compare } \sin \frac{x}{2}: R \cos \alpha = 5 \quad (1)$$

$$\text{Compare } \cos \frac{x}{2}: R \sin \alpha = 12 \quad (2)$$

$$\text{Divide (2) by (1): } \tan \alpha = \frac{12}{5}$$

$$\Rightarrow \alpha = 67.38^\circ \text{ (2 d.p.)}$$

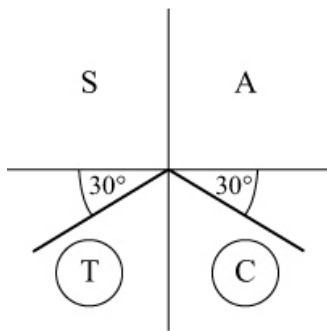
$$R = 13$$

$$\text{Solve } 13 \sin \left(\frac{x}{2} - 67.38^\circ \right) = -6.5,$$

in the interval $-360^\circ \leq x \leq 360^\circ$

$$\text{So } \sin \left(\frac{x}{2} - 67.38^\circ \right) = -\frac{1}{2},$$

$$-247.4^\circ \leq \frac{x}{2} - 67.4^\circ \leq 112.6^\circ$$



From quadrant diagram:

$$\frac{x}{2} - 67.38^\circ = -150^\circ, -30^\circ$$

$$\Rightarrow \frac{x}{2} = -82.62^\circ, 37.38^\circ$$

$$\Rightarrow x = -165.2^\circ, 74.8^\circ$$

$$\begin{aligned}
 9 \text{ a} \quad & \text{Set } 3 \sin 3\theta - 4 \cos 3\theta \equiv R \sin(3\theta - \alpha) \\
 & 3 \sin 3\theta - 4 \cos 3\theta \\
 & \equiv R \sin 3\theta \cos \alpha - R \cos 3\theta \sin \alpha
 \end{aligned}$$

$$\text{Compare } \sin 3\theta: R \cos \alpha = 3 \quad (1)$$

$$\text{Compare } \cos 3\theta: R \sin \alpha = 4 \quad (2)$$

$$\text{Divide (2) by (1): } \tan \alpha = \frac{4}{3}$$

$$\Rightarrow \alpha = 53.13^\circ \text{ (2 d.p.)}$$

$$R^2 = 3^2 + 4^2 = 25 \Rightarrow R = 5$$

$$\text{So } 3 \sin 3\theta - 4 \cos 3\theta = 5 \sin(3\theta - 53.13^\circ)$$

b The minimum value of $3 \sin 3\theta - 4 \cos 3\theta$ is -5 . This occurs when

$$\sin(3\theta - 53.13^\circ) = -1$$

$$3\theta - 53.13^\circ = 270^\circ$$

$$\theta = 107.7^\circ$$

c $5 \sin(3\theta - 53.13^\circ) = 1$,
in the interval $0 \leq \theta < 180^\circ$

$$\text{So } \sin(3\theta - 53.13^\circ) = \frac{1}{5},$$

in the interval

$$-53.13^\circ \leq 3\theta - 53.13^\circ < 506.87^\circ$$

$$3\theta - 53.13^\circ = 11.54^\circ, 168.46^\circ, 371.54^\circ$$

$$\theta = 21.6^\circ, 73.9^\circ, 141.6^\circ$$

$$10 \text{ a} \quad \text{As } \sin^2 \theta = \frac{1 - \cos 2\theta}{2} \text{ and}$$

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$$

$$\text{So } 5 \sin^2 \theta - 3 \cos^2 \theta + 6 \sin \theta \cos \theta$$

$$\equiv 5 \frac{1 - \cos 2\theta}{2} - 3 \frac{1 + \cos 2\theta}{2}$$

$$+ 3(2 \sin \theta \cos \theta)$$

$$\equiv \frac{5}{2} - \frac{5}{2} \cos 2\theta - \frac{3}{2} - \frac{3}{2} \cos 2\theta + 3 \sin 2\theta$$

$$\equiv 1 - 4 \cos 2\theta + 3 \sin 2\theta$$

- 10 b** Write $3\sin 2\theta - 4\cos 2\theta$ in the form $R\sin(2\theta - \alpha)$
 The maximum value of $R\sin(2\theta - \alpha)$ is R
 The minimum value of $R\sin(2\theta - \alpha)$ is $-R$

You know that $R^2 = 3^2 + 4^2$ so $R = 5$

So maximum value of

$$1 - 4\cos 2\theta + 3\sin 2\theta \text{ is } 1 + 5 = 6$$

and minimum value of

$$1 - 4\cos 2\theta + 3\sin 2\theta \text{ is } 1 - 5 = -4$$

- c** $1 - 4\cos 2\theta - 3\sin 2\theta = -1$
 $\Rightarrow 3\sin 2\theta - 4\cos 2\theta = -2$

Write $3\sin 2\theta - 4\cos 2\theta$ in the form $R\sin(2\theta - \alpha)$

So $R\sin(2\theta - \alpha) = -2$

$$\Rightarrow 5\sin(2\theta - 53.13^\circ) = -2$$

(By solving in same way as Question 9, part a)

Look for solutions in the interval $-53.13^\circ \leq 2\theta - 53.13^\circ < 306.87^\circ$

$$2\theta - 53.13^\circ = -23.58, 203.58$$

$$\theta = 14.8^\circ, 128.4^\circ$$

- 11 a** Let $3\cos\theta + \sin\theta \equiv R\cos(\theta - \alpha)$
 $\equiv R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

Compare $\cos\theta$: $R\cos\alpha = 3$ (1)

Compare $\sin\theta$: $R\sin\alpha = 1$ (2)

Divide (2) by (1): $\tan\alpha = \frac{1}{3}$

$$\Rightarrow \alpha = 18.43^\circ \text{ (2 d.p.)}$$

$$R^2 = 3^2 + 1^2 = 10 \Rightarrow R = \sqrt{10} = 3.16$$

Solve $\sqrt{10}\cos(\theta - 18.43^\circ) = 2$,

in the interval $0 \leq \theta \leq 360^\circ$

$$\Rightarrow \cos(\theta - 18.43^\circ) = \frac{2}{\sqrt{10}}$$

$$\Rightarrow \theta - 18.43^\circ = 50.77^\circ, 309.23^\circ$$

$$\Rightarrow \theta = 69.2^\circ, 327.7^\circ$$

- b** Squaring $3\cos\theta = 2 - \sin\theta$
 gives $9\cos^2\theta = 4 + \sin^2\theta - 4\sin\theta$
 $\Rightarrow 9(1 - \sin^2\theta) = 4 + \sin^2\theta - 4\sin\theta$
 $\Rightarrow 10\sin^2\theta - 4\sin\theta - 5 = 0$

- c** $10\sin^2\theta - 4\sin\theta - 5 = 0$

$$\Rightarrow \sin\theta = \frac{4 \pm \sqrt{216}}{20}$$

For $\sin\theta = \frac{4 + \sqrt{216}}{20}$, $\sin\theta$ is positive,

so θ is in the first and second quadrants.

$$\Rightarrow \theta = 69.2^\circ, 180^\circ - 69.2^\circ$$

$$= 69.2^\circ, 110.8^\circ$$

For $\sin\theta = \frac{4 - \sqrt{216}}{20}$, $\sin\theta$ is negative,

so θ is in the third and fourth quadrants.

$$\Rightarrow \theta = 180^\circ - (-32.3^\circ), 360^\circ + (-32.3^\circ)$$

$$= 212.3^\circ, 327.7^\circ$$

So solutions of quadratic in (b) are

$$69.2^\circ, 110.8^\circ, 212.3^\circ, 327.7^\circ$$

- d** In squaring the equation, you are also including the solutions to

$$3\cos\theta = -(2 - \sin\theta),$$

which when squared produces the same quadratic. The extra two solutions satisfy this equation.

$$12 \text{ a } \cot \theta + 2 = \operatorname{cosec} \theta$$

$$\Rightarrow \frac{\cos \theta}{\sin \theta} + 2 = \frac{1}{\sin \theta}$$

Multiplying both sides by $\sin \theta$ gives

$$\cos \theta + 2 \sin \theta = 1$$

$$12 \text{ b } \cos \theta + 2 \sin \theta = 1$$

$$\begin{aligned} \text{Set } 2 \sin \theta + \cos \theta &\equiv R \sin(\theta + \alpha) \\ &\equiv R \sin \theta \cos \alpha + R \cos \theta \sin \alpha \end{aligned}$$

$$\text{So } R \cos \alpha = 1 \text{ and } R \sin \alpha = 1$$

$$\frac{R \sin \alpha}{R \cos \alpha} = \tan \alpha = \frac{1}{2}$$

$$\alpha = \tan^{-1} \frac{1}{2} = 26.57^\circ \text{ (2 d.p.)}$$

$$R^2 = 2^2 + 1^2 \Rightarrow R = \sqrt{5}$$

$$\text{So } \sqrt{5} \sin(\theta + 26.57^\circ) = 1$$

$$\sin(\theta + 26.57^\circ) = \frac{1}{\sqrt{5}}, \text{ in the interval}$$

$$26.57^\circ \leq \theta + 26.57^\circ < 386.57^\circ$$

$$\theta + 26.57 = 26.57, 153.43$$

$$\theta = 0^\circ, 126.9^\circ$$

As both $\cot \theta$ and $\operatorname{cosec} \theta$ are undefined at 0, $\theta = 126.9^\circ$ is the only solution.

$$13 \text{ a } \sqrt{2} \cos\left(\theta - \frac{\pi}{4}\right) + (\sqrt{3} - 1) \sin \theta = 2$$

$$\begin{aligned} \Rightarrow \sqrt{2} \cos \theta \cos \frac{\pi}{4} + \sqrt{2} \sin \theta \sin \frac{\pi}{4} \\ + \sqrt{3} \sin \theta - \sin \theta = 2 \end{aligned}$$

$$\begin{aligned} \Rightarrow \sqrt{2} \left(\frac{1}{\sqrt{2}} \cos \theta + \frac{1}{\sqrt{2}} \sin \theta \right) \\ + \sqrt{3} \sin \theta - \sin \theta = 2 \end{aligned}$$

$$\Rightarrow \cos \theta + \sin \theta + \sqrt{3} \sin \theta - \sin \theta = 2$$

$$\Rightarrow \cos \theta + \sqrt{3} \sin \theta = 2$$

$$12 \text{ b } \cos \theta + \sqrt{3} \sin \theta = 2$$

$$\begin{aligned} \text{Set } \sqrt{3} \sin \theta + \cos \theta &\equiv R \sin(\theta + \alpha) \\ &\equiv R \sin \theta \cos \alpha + R \cos \theta \sin \alpha \end{aligned}$$

$$\text{So } R \cos \alpha = \sqrt{3} \text{ and } R \sin \alpha = 1$$

$$\frac{R \sin \alpha}{R \cos \alpha} = \tan \alpha = \frac{1}{\sqrt{3}}$$

$$\alpha = \tan^{-1} \left(\frac{1}{\sqrt{3}} \right) = \frac{\pi}{6}$$

$$R^2 = \sqrt{3}^2 + 1^2 = 4 \Rightarrow R = 2$$

$$2 \sin \left(\theta + \frac{\pi}{6} \right) = 2$$

$$\sin \left(\theta + \frac{\pi}{6} \right) = 1, \text{ in the interval}$$

$$\frac{\pi}{6} \leq \theta + \frac{\pi}{6} \leq \frac{11\pi}{6}$$

$$\theta + \frac{\pi}{6} = \frac{\pi}{2}$$

$$\theta = \frac{\pi}{3}$$

$$14 \text{ a } \text{Set } 9 \cos \theta + 40 \sin \theta \equiv R \cos(\theta - \alpha) \\ \equiv R \cos \theta \cos \alpha + R \sin \theta \sin \alpha$$

$$\text{So } R \cos \alpha = 9 \text{ and } R \sin \alpha = 40$$

$$\frac{R \sin \alpha}{R \cos \alpha} = \tan \alpha = \frac{40}{9}$$

$$\alpha = \tan^{-1} \left(\frac{40}{9} \right)$$

$$\text{So } \alpha = 77.320^\circ$$

$$R^2 \cos^2 \alpha + R^2 \sin^2 \alpha = 40^2 + 9^2$$

$$R^2 (\cos^2 \alpha + \sin^2 \alpha) = 1681$$

$$R = 41$$

$$\text{So } 9 \cos \theta + 40 \sin \theta = 41 \cos(\theta - 77.320^\circ)$$

$$14 \text{ b i } g(\theta) = \frac{18}{50 + 41 \cos(\theta - 77.320^\circ)}$$

The minimum value of $g(\theta)$ is when $\cos(\theta - 77.320^\circ) = 1$

$$\text{So the minimum value is } \frac{18}{50 + 41} = \frac{18}{91}$$

14 b ii The minimum occurs when

$$\cos(\theta - 77.320^\circ) = 1$$

$$\theta - 77.320^\circ = 0$$

$$\theta = 77.320^\circ$$

15 a Set $12 \cos 2\theta - 5 \sin 2\theta \equiv R \cos(2\theta + \alpha)$
 $\equiv R \cos 2\theta \cos \alpha - R \sin 2\theta \sin \alpha$

So $R \cos \alpha = 12$ and $R \sin \alpha = 5$

$$\frac{R \sin \alpha}{R \cos \alpha} = \tan \alpha = \frac{5}{12}$$

$$\alpha = \tan^{-1} \frac{5}{12}$$

So $\alpha = 22.62^\circ$ (2 d.p.)

$$R^2 \cos^2 \alpha + R^2 \sin^2 \alpha = 12^2 + 5^2$$

$$R^2 (\cos^2 \alpha + \sin^2 \alpha) = 169$$

$$R^2 = 169$$

$$R = 13 \text{ and } \alpha = 22.6^\circ$$

b $13 \cos(2\theta + 22.62^\circ) = -6.5$

$$\cos(2\theta + 22.62^\circ) = -\frac{6.5}{13}, \text{ in the interval}$$

$$22.62^\circ \leq 2\theta + 22.62^\circ < 382.62^\circ$$

$$2\theta + 22.62^\circ = 120^\circ, 240^\circ$$

$$\theta = 48.7^\circ, 108.7^\circ$$

c $24 \cos^2 \theta - 10 \sin \theta \cos \theta$
 $\equiv 24 \left(\frac{\cos 2\theta + 1}{2} \right) - 5 \sin 2\theta$

$$\equiv 12 \cos 2\theta - 5 \sin 2\theta + 12$$

So $a = 12$, $b = -5$ and $c = 12$

d $24 \cos^2 \theta - 10 \sin \theta \cos \theta$
 $\equiv 12 \cos 2\theta - 5 \sin 2\theta + 12$

From part (a)

$$12 \cos 2\theta - 5 \sin 2\theta + 12$$

$$= 13 \cos(2\theta + 22.62^\circ) + 12$$

The minimum value is therefore when

$$\cos(2\theta + 22.62^\circ) = -1$$

It is $13(-1) + 12 = -1$